Contemporary Mathematics

# What is a Toric Variety?

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Abstract. This paper is a tutorial in the basic theory of toric varieties. It discusses their definition using fans, homogeneous coordinates, and polytopes. Numerous examples are included.

## Introduction

Toric varieties were first defined in the 1970s and have become an important part of algebraic geometry. They can be used in many different geometric situations yet also have interesting connections with combinatorics and convex polytopes.

This article is an introduction to toric varieties for non-specialists. Many examples are given to illustrate the various definitions. The paper is organized into 14 sections as follows:



## 1. Varieties

We will work over the complex numbers $\mathbb{C}$. Basic examples of varieties are:

- Affine space $\mathbb{C}^n$ and *affine varieties*

$$V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$$

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defined by the polynomial equations $f_1 = \cdots = f_s = 0$.

- Projective space $\mathbb{P}^n$ and *projective varieties*
$$V = \mathbf{V}(F_1, \ldots, F_s) \subset \mathbb{P}^n$$
defined by the homogeneous equations $F_1 = \cdots = F_s = 0$.

In this article, most varieties will be either affine or projective.

EXAMPLE 1.1. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\} = \{z \in \mathbb{C} \mid z \neq 0\}$. Then $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ is an affine variety since the map $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 1/t_1 \cdots t_n)$ gives
$$(\mathbb{C}^*)^n \simeq \mathbf{V}(x_1 x_2 \cdots x_{n+1} - 1) \subset \mathbb{C}^{n+1}.$$
In the theory of algebraic groups, $(\mathbb{C}^*)^n$ is called the *$n$-dimensional complex torus*. This is where the "toric" in "toric variety" comes from. □

Given varieties $W \subset V$, we call the complement $V \setminus W = \{v \in V \mid v \notin W\}$ a *Zariski open subset* of $V$. These are the points of $V$ where one or more of the defining equations of $W$ don't vanish.

EXAMPLE 1.2. Notice that
$$(\mathbb{C}^*)^n \subset \mathbb{C}^n$$
is a Zariski open since $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1 \cdots x_n)$. □

A variety $V$ is *irreducible* if it cannot be written as union $V = V_1 \cup V_2$ where $V_1 \neq V$ and $V_2 \neq V$ are varieties.

## 2. Toric Varieties

DEFINITION 2.1. *A* ***toric variety*** *is an irreducible variety $V$ such that*

(1) *$(\mathbb{C}^*)^n$ is a Zariski open subset of $V$, and*
(2) *the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $V$.*

We will see later that the theory of toric varieties works best when $V$ is *normal* (we defer the definition of normality since it is somewhat technical). Here are the most basic examples of toric varieties.

EXAMPLE 2.2. $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are clearly toric varieties. As for $\mathbb{P}^n$, suppose that $x_0, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$. The map
$$(\mathbb{C}^*)^n \longrightarrow \mathbb{P}^n$$
defined by $(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n)$ allows us to identify $(\mathbb{C}^*)^n$ with the Zariski open subset $\mathbb{P}^n \setminus \mathbf{V}(x_0 x_1 \cdots x_n)$. Then setting
$$(t_1, \ldots, t_n) \cdot (a_0, a_1, \ldots, a_n) = (a_0, t_1 a_1, \ldots, t_n a_n)$$
shows that $\mathbb{P}^n$ is a toric variety. □

In studying toric varieties, points $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ play two important roles:

- First, a *Laurent monomial* is defined by
$$\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_n^{a_n}.$$
Note that $\mathbf{t}^{\mathbf{a}}$ gives a function $(\mathbb{C}^*)^n \to \mathbb{C}^*$. In the theory of algebraic groups, this is called a *character*. The $\mathbb{C}$-linear span of all Laurent monomials is the ring $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ of *Laurent polynomials*.

- Second, a 1-*parameter subgroup* $\lambda^{\mathbf{a}} : \mathbb{C}^* \to (\mathbb{C}^*)^n$ is defined by

$$\lambda^{\mathbf{a}}(t) = (t^{a_1}, \dots, t^{a_n}).$$

In general, a toric variety $V$ consists of $(\mathbb{C}^*)^n$ plus some "extra stuff." When $V$ is affine, we will see that the "extra stuff" is determined by which Laurent monomials $\mathbf{t}^{\mathbf{m}}$ are defined on $V$. Here is an example.

EXAMPLE 2.3. Consider the toric variety $\mathbb{C}^n$. The one easily sees that the Laurent monomial $\mathbf{t}^{\mathbf{m}} = t_1^{m_1} \cdots t_n^{m_n}$ determined by $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ extends to a function $\mathbb{C}^n \to \mathbb{C}$ if and only if $m_i \geq 0$ for all $i$. Below we will construct $\mathbb{C}^n$ using only these Laurent monomials in $\mathbb{Z}^n$. □

## 3. Examples of Toric Varieties

Besides the basic examples of toric varieties given above, we also have the following.

EXAMPLE 3.1. If $V$ and $W$ are toric varieties, then so is $V \times W$. This shows, for instance, that $\mathbb{P}^1 \times \mathbb{P}^1$ is a toric variety. □

EXAMPLE 3.2. Consider the cuspidal cubic $C = \mathbf{V}(y^2 - x^3) \subset \mathbb{C}^2$. This contains $\mathbb{C}^*$ via the map $t \mapsto (t^2, t^3)$, and $\mathbb{C}^*$ acts on $C$ via $t \cdot (u, v) = (t^2 u, t^3 v)$. □

The previous example is interesting because it is a non-normal toric variety. In dimension one, the only normal toric varieties are $\mathbb{C}^*$, $\mathbb{C}$ and $\mathbb{P}^1$.

EXAMPLE 3.3. Consider $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. This contains the torus $(\mathbb{C}^*)^3$ via the map

$$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}).$$

Question: Which Laurent monomials $\mathbf{t}^{\mathbf{m}}$ extend to functions $V \to \mathbb{C}$? If $\mathbf{m} = (a, b, c) \in \mathbb{Z}^3$, then we get the function on $V$ defined by $x^a y^b z^c$. If $a, b, c \geq 0$, then this certainly extends. However, suppose that $c < 0$ and $a + c, b + c \geq 0$. Then, since $xy = zw$ on $V$, we have

$$x^a y^b z^c = x^a y^b \left(\frac{xy}{w}\right)^c = x^{a+c} y^{b+c} w^{-c},$$

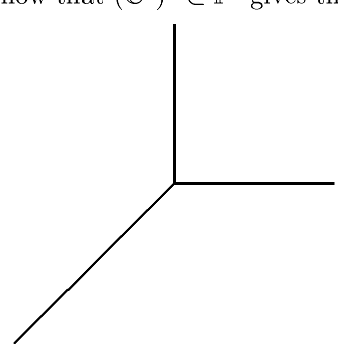
which shows that $\mathbf{t}^{\mathbf{m}}$ extends to a function $V \to \mathbb{C}$. We will see below that the inequalities

$$a \geq 0,\ b \geq 0,\ a + c \geq 0,\ b + c \geq 0 \tag{3.1}$$

define the *dual cone* corresponding to the normal affine toric variety $V$. □

EXAMPLE 3.4. Let's show that $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$ gives the following picture in $\mathbb{R}^2$:

(3.2)



A 1-parameter subgroup $\mathbf{u} \in \mathbb{Z}^2$ gives a map $\lambda^{\mathbf{u}} : \mathbb{C}^* \to \mathbb{P}^2$. Since $\mathbb{P}^2$ is compact, the limit

$$\lim_{t \to 0} \lambda^{\mathbf{u}}(t)$$

exists in $\mathbb{P}^2$. If $\mathbf{u} = (a, b) \in \mathbb{Z}^2$, then

$$\lambda^{\mathbf{u}}(t) = (1, t^a, t^b).$$

It is then straightforward to compute that

$$\lim_{t \to 0} \lambda^{\mathbf{u}}(t) = \lim_{t \to 0} (1, t^a, t^b) = \begin{cases} (1,0,0) & a, b > 0 \\ (1,0,1) & a > 0, b = 0 \\ (1,1,0) & a = 0, b > 0 \\ (1,1,1) & a = b = 0 \\ (0,0,1) & a > b, b < 0 \\ (0,1,0) & a < 0, a < b \\ (0,1,1) & a < 0, a = b. \end{cases} \tag{3.3}$$

The first four cases are trivial. To see how the fifth case works, note that

$$\lim_{t \to 0} (1, t^a, t^b) = \lim_{t \to 0} (t^{-b}, t^{a-b}, 1)$$

since these are homogeneous coordinates. Then $a > b$ and $b < 0$ imply that the limit is $(0, 0, 1)$, as claimed. The last two cases are similar.

Now observe that (3.2) decomposes the plane into 7 disjoint regions:

- The open sets $a, b > 0$; $a < 0, a < b$; and $a > b, b < 0$.
- The open rays $a > 0, b = 0$; $a = 0, b > 0$; and $a < 0, a = b$.
- The point $a = b = 0$.

The corresponds perfectly with (3.3). We will see below that (3.2) is the *fan* corresponding to the toric variety $\mathbb{P}^2$. □

## 4. Cones

A *rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of $\mathbb{Z}^n$:

$$\sigma = \{\lambda_1 \mathbf{u}_1 + \cdots + \lambda_\ell \mathbf{u}_\ell \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_\ell \geq 0\},$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_\ell \in \mathbb{Z}^n$. Then:

- $\sigma$ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.
- The *dimension* of $\sigma$ is the dimension of the smallest subspace of $\mathbb{R}^n$ containing $\sigma$.
- A *face* of $\sigma$ is the intersection $\{\ell = 0\} \cap \sigma$, where $\ell$ is a linear form which is nonnegative on $\sigma$.
- The *edges* of $\sigma$ are its 1-dimensional faces. Edges are denoted by $\rho$. The *primitive element* $\mathbf{n}_\rho$ of an edge $\rho$ is the unique generator of $\rho \cap \mathbb{Z}^n$. The cone $\sigma$ is generated by the primitive elements $\mathbf{n}_\rho$ of its edges $\rho$.
- The *facets* of $\sigma$ are its codimension-1 faces. When $\dim \sigma = n$, each facet has an inward pointing normal which is an element of $\mathbb{R}^n$. We get a unique inward normal by requiring that it is in $\mathbb{Z}^n$ and has minimal length.

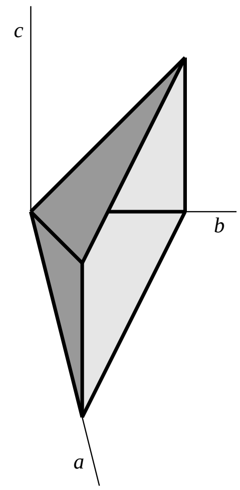
DEFINITION 4.1. *If $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone, then its **dual cone** $\sigma^\vee \subset \mathbb{R}^n$ is*

$$\sigma^\vee = \{\mathbf{m} \in \mathbb{R}^n \mid \langle \mathbf{m}, \mathbf{u} \rangle \geq 0 \text{ for all } \mathbf{u} \in \sigma\},$$

*where $\langle \mathbf{m}, \mathbf{u} \rangle$ is the usual dot product on $\mathbb{R}^n$. This is a rational polyhedral cone of dimension $n$.*

Here is an example of a cone and its dual.

EXAMPLE 4.2. Consider the cone $\sigma \subset \mathbb{R}^3$ pictured below:




This cone is generated by the primitive elements

$$\mathbf{n}_1 = (1,0,0),\ \mathbf{n}_2 = (0,1,0),\ \mathbf{n}_3 = (1,0,1),\ \mathbf{n}_4 = (0,1,1) \tag{4.1}$$

in $\mathbb{Z}^3$, and the inward pointing normals of the facets of $\sigma$ are

$$\mathbf{m}_1 = (1,0,0),\ \mathbf{m}_2 = (0,1,0),\ \mathbf{m}_3 = (0,0,1),\ \mathbf{m}_4 = (1,1,-1) \tag{4.2}$$

in $\mathbb{Z}^3$. It follows that these generate the dual cone $\sigma^\vee$ in $\mathbb{R}^3$. Thus $(a,b,c) \in \sigma^\vee$ if and only if

$$a \geq 0,\ b \geq 0,\ a+c \geq 0,\ b+c \geq 0$$

These are precisely the inequalities (3.1). □

## 5. Affine Toric Varieties

Let $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone with dual cone $\sigma^\vee \subset \mathbb{R}^n$. Our goal is to show that this determines a normal affine toric variety $U_\sigma$. The basic idea is as follows. We call $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ a *lattice point* of $\sigma^\vee$. Each lattice point $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ gives a Laurent monomial $\mathbf{t}^{\mathbf{m}}$. Then $U_\sigma$ should be the the smallest variety on which these Laurent monomials are defined everywhere.

We will construct $U_\sigma$ using *Gordan's Lemma*, which implies that $\sigma^\vee \cap \mathbb{Z}^n$ is finitely generated. In other words, there are $\mathbf{m}_1, \ldots, \mathbf{m}_\ell \in \sigma^\vee \cap \mathbb{Z}^n$ such that every element of $\sigma^\vee \cap \mathbb{Z}^n$ is of the form

$$a_1\mathbf{m}_1 + \cdots + a_\ell\mathbf{m}_\ell,\ a_i \in \mathbb{Z},\ a_i \geq 0. \tag{5.1}$$

The generators $\mathbf{m}_1, \ldots, \mathbf{m}_\ell$ determine the affine variety $U_\sigma \subset \mathbb{C}^\ell$ as follows. Consider

$$\varphi : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^\ell \tag{5.2}$$

defined by

$$\varphi(t_1, \ldots, t_n) = \big(\mathbf{t}^{\mathbf{m}_1}(t_1, \ldots, t_n), \ldots, \mathbf{t}^{\mathbf{m}_\ell}(t_1, \ldots, t_n)\big).$$

Then $U_\sigma \subset \mathbb{C}^\ell$ is the *Zariski closure* of the image of this map. This means that $U_\sigma$ is the smallest variety containing the image of (5.2).

One can prove that the map $(\mathbb{C}^*)^n \to U_\sigma$ induced by (5.2) is an inclusion and makes $U_\sigma$ into a toric variety. Furthermore:

- By (5.2), $\mathbf{t}^{\mathbf{m}_i}$ extends to the function $U_\sigma \to \mathbb{C}$ given by the projection of $U_\sigma \subset \mathbb{C}^\ell$ onto the $i$th coordinate. Thus $\mathbf{t}^{\mathbf{m}_i}$ is defined on all of $U_\sigma$.
- Since every $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ is of the form (5.1), it follows that $\mathbf{t}^{\mathbf{m}}$ extends to a function on $U_\sigma$.
- $U_\sigma$ is the smallest variety where the $\mathbf{t}^{\mathbf{m}}$ are defined since it is the Zariski closure of (5.2).

We say that $U_\sigma$ is the *normal affine toric variety* determined by the strictly convex rational polyhedral cone $\sigma$. Normality will be explained in Section 7.

Here is an easy example.

EXAMPLE 5.1. First consider the $n$-dimensional cone $\sigma$ generated by the standard basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$. Thus $\sigma$ is the "first orthant" of $\mathbb{R}^n$ where all coordinates are nonnegative. Then $\sigma^\vee$ has the same description in $\mathbb{R}^n$, so that $e_1, \ldots, e_n$ generate $\sigma^\vee \cap \mathbb{Z}^n$ over $\mathbb{Z}_{\geq 0}$. Since $\mathbf{t}^{e_i} = t_i$, it follows that (5.2) is the inclusion $(\mathbb{C}^*)^n \subset \mathbb{C}^n$. This gives $U_\sigma = \mathbb{C}^n$. □

By (5.2), $U_\sigma \subset \mathbb{C}^\ell$ is the variety of $\mathbb{C}^\ell$ whose defining equations are determined by the algebraic relations among the $\mathbf{t}^{\mathbf{m}_i}$. Here is an example to illustrate what this means.

EXAMPLE 5.2. Consider the cone $\sigma \subset \mathbb{R}^3$ pictured in Example 4.2. It is easy to see that that the generators of $\sigma^\vee \cap \mathbb{Z}^3$ are the vectors

$$\mathbf{m}_1 = (1,0,0),\ \mathbf{m}_2 = (0,1,0),\ \mathbf{m}_3 = (0,0,1),\ \mathbf{m}_4 = (1,1,-1).$$

from (4.2). Thus (5.2) is defined by

$$(t_1, t_2, t_3) \mapsto (\mathbf{t}^{\mathbf{m}_1}, \mathbf{t}^{\mathbf{m}_2}, \mathbf{t}^{\mathbf{m}_3}, \mathbf{t}^{\mathbf{m}_4}) = (t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \in \mathbb{C}^4. \tag{5.3}$$

If $x, y, z, w$ are variables on $\mathbb{C}^4$, then $\mathbf{t}^{\mathbf{m}_1}\mathbf{t}^{\mathbf{m}_2} = t_1 t_2 = t_3(t_1 t_2 t_3^{-1}) = \mathbf{t}^{\mathbf{m}_3}\mathbf{t}^{\mathbf{m}_4}$ implies that $xy - zw$ vanishes on the image of (5.3). It follows that $U_\sigma \subset \mathbf{V}(xy - zw)$, and in fact, one can show that

$$U_\sigma = \mathbf{V}(xy - zw) \subset \mathbb{C}^4.$$

This gives the toric variety from Example 3.3.

The vanishing of $xy - zw$ on $U_\sigma$ follows from the relation $\mathbf{m}_1 + \mathbf{m}_2 = \mathbf{m}_3 + \mathbf{m}_4$ between the generators of $\sigma^\vee \cap \mathbb{Z}^3$. Thus the ideal $\langle xy - zw \rangle$ defining $U_\sigma$ is determined by the integer linear relations among the $\mathbf{m}_i$. This is true in general and is related to the theory of *toric ideals* to be discussed in the article [24] by Frank Sottile in this volume. □

## 6. Coordinate Rings

In algebraic geometry, the ring of polynomial functions on an affine variety is called the *coordinate ring* of the affine variety. For example, the coordinate ring of $\mathbb{C}^n$ is $\mathbb{C}[x_1, \ldots, x_n]$.

For an affine toric variety $U_\sigma$, we can give an especially nice description of the coordinate ring. Namely, each $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ gives the Laurent monomial $\mathbf{t}^{\mathbf{m}} \in \mathbb{C}[t_1, t_1{-}1, \ldots, t_n, t_n^{-1}]$. Then consider

$$\text{(6.1)} \qquad \operatorname{Span}(\mathbf{t}^{\mathbf{m}} \mid \mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n) \subset \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}].$$

This is a ring since $\mathbf{m}, \mathbf{m}' \in \sigma^\vee \cap \mathbb{Z}^n$ implies $\mathbf{m} + \mathbf{m}' \in \sigma^\vee \cap \mathbb{Z}^n$, so that if $\mathbf{t}^{\mathbf{m}}$ and $\mathbf{t}^{\mathbf{m}'}$ are in (6.1), then the product $\mathbf{t}^{\mathbf{m}}\mathbf{t}^{\mathbf{m}'} = \mathbf{t}^{\mathbf{m}+\mathbf{m}'}$ is too.

In the language of semigroup algebras, the ring (6.1) is denoted

$$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n].$$

This is the notation used in the literature on toric varieties. The previous section shows that every Laurent monomial in this ring gives a polynomial function on $U_\sigma$ and hence lies in the coordinate ring of $U_\sigma$. In fact, one can prove that

$$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] = \text{the coordinate ring of } U_\sigma.$$

Also note that if $\mathbf{m}_1, \ldots, \mathbf{m}_\ell \in \sigma^\vee \cap \mathbb{Z}^n$ generate $\sigma^\vee \cap \mathbb{Z}^n$ in the sense of (5.1), then

$$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] = \mathbb{C}[\mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_\ell}] \subset \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}].$$

Thus the coordinate ring consists of all polynomial expressions in the Laurent monomials $\mathbf{t}^{\mathbf{m}_i}$. Here is an example.

Example 6.1. For the cone $\sigma$ of Example 5.2, the Laurent monomials appearing in (5.3) show that

$$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^3] = \mathbb{C}[t_1, t_2, t_3, t_1 t_2 t_3^{-1}] \subset \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}].$$

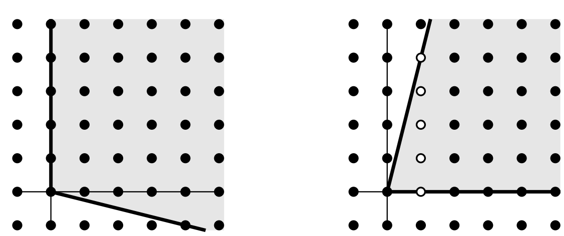
This gives an explicit representation of the coordinate ring of $U_\sigma$ in this case. □

## 7. Normality

A variety is *normal* if its local rings are integrally closed in their fields of fractions. This definition is unlikely to be helpful to the nonexpert. Our goal here is to describe what normality means for affine toric varieties. The key point is that the affine toric variety $U_\sigma$ defined in the previous section is always normal.

To motivate our discussion, consider the following example.

Example 7.1. Here is a cone and its dual:



The cone $\sigma$ The cone $\sigma^\vee$

The generators of $\sigma^\vee \cap \mathbb{Z}^2$ are $\mathbf{m}_i = (1, i)$ for $i = 0, \ldots, 4$. It follows that $U_\sigma \subset \mathbb{C}^5$ is the Zariski closure of the image of the parametrization $(\mathbb{C}^*)^2 \to \mathbb{C}^5$ defined by

$$(t, u) \mapsto (t, tu, tu^2, tu^3, tu^4). \tag{7.1}$$

What happens if we only use some of these monomials? Here are two things which can occur.

First, suppose we use only $\mathbf{m}_0 = (1, 0)$ and $\mathbf{m}_4 = (1, 4)$. Over $\mathbb{R}_{\geq 0}$, these generate $\sigma^\vee$ and give the map $(\mathbb{C}^*)^2 \to \mathbb{C}^2$ defined by

$$(t, u) \mapsto (t, tu^4). \tag{7.2}$$

The Zariski closure of the image is $\mathbb{C}^2$, but (7.2) is 4-to-1. One can show that this happens because $\mathbf{m}_0$ and $\mathbf{m}_4$ do not generate $\mathbb{Z}^2$ over $\mathbb{Z}$. The point is, an affine toric variety involves both a cone and a lattice. So $\mathbf{m}_0$ and $\mathbf{m}_4$ don't work because they mess up the lattice, even though they do generate the dual cone.

Second, suppose we use $\mathbf{m}_0 = (1, 0)$, $\mathbf{m}_1 = (1, 1)$ and $\mathbf{m}_4 = (1, 4)$. They generate the dual cone over $\mathbb{R}_{\geq 0}$ and give the map $(\mathbb{C}^*)^2 \to \mathbb{C}^3$ defined by

$$(t, u) \mapsto (t, tu, tu^4). \tag{7.3}$$

This map is 1-to-1, which is easy to see directly and also because $\mathbf{m}_0$, $\mathbf{m}_1$ and $\mathbf{m}_4$ generate the lattice $\mathbb{Z}^2$. However, one can compute that the Zariski closure of the image of (7.3) is $y^4 = x^3 z$. It is also straightforward to show that the singular locus of this surface is the line $x = y = 0$. Since the singular locus of a normal variety has codimension at least 2, it follows that this variety is not normal. Thus we have an example of a non-normal toric variety. $\square$

In the above example, the normal toric variety $U_\sigma$ determined by the cone $\sigma$ and lattice $\mathbb{Z}^2$ was constructed as the Zariski closure of the map (7.1) from $(\mathbb{C}^*)^2$ to $\mathbb{C}^5$. Then (7.2) and (7.3) are other toric varieties obtained by projecting $U_\sigma$ to $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. These projections are not the normal toric variety for $\sigma$ and $\mathbb{Z}^2$, because:

- In (7.2), we kept the dual cone but changed the lattice.
- In (7.3), we kept the lattice and the dual cone, but lost normality.

As we will see below, the key reason for the second bullet is that $\mathbf{m}_0$, $\mathbf{m}_1$ and $\mathbf{m}_4$ do not generate $\sigma^\vee \cap \mathbb{Z}^2$ over $\mathbb{Z}_{\geq 0}$.

To generalize this example, let $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone, and suppose that we have $\widetilde{\mathbf{m}}_i \in \sigma^\vee \cap \mathbb{Z}^n$ for $i = 1, \ldots, s$. Then, using the $\mathbf{t}^{\widetilde{\mathbf{m}}_i}$ as in (5.2), we get a map

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{C}^s. \tag{7.4}$$

THEOREM 7.2. *The Zariski closure of the image of* (7.4) *is the normal affine toric variety* $U_\sigma$ *determined by* $\sigma$ *and* $\mathbb{Z}^n$ *if and only if* $\sigma^\vee \cap \mathbb{Z}^n$ *is generated over* $\mathbb{Z}_{\geq 0}$ *by* $\widetilde{\mathbf{m}}_i$ *for* $i = 1, \ldots, s$.

Thus an affine toric variety is normal precisely when you use *all* lattice points in the dual cone.

## 8. Fans and Toric Varieties

We next create more general normal toric varieties by gluing together affine toric varieties containing the same torus $(\mathbb{C}^*)^n$. This brings us to the concept of a *fan*, which is defined to be a finite collection $\Sigma$ of cones in $\mathbb{R}^n$ with the following three properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and $\tau$ is a face of $\sigma$, then $\tau \in \Sigma$.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

Each $\sigma \in \Sigma$ gives an affine toric variety $U_\sigma$, and if $\tau$ is a face of $\sigma$, then $U_\tau$ can be regarded as a Zariski open subset of $U_\sigma$. This leads to the following definiton.

DEFINITION 8.1. *Given a fan $\Sigma$ in $\mathbb{R}^n$, $X_\Sigma$ is the variety obtained from the affine varieties $U_\sigma$, $\sigma \in \Sigma$, by gluing together $U_\sigma$ and $U_\tau$ along their common open subset $U_{\sigma\cap\tau}$ for all $\sigma, \tau \in \Sigma$.*

The inclusions $(\mathbb{C}^*)^n \subset U_\sigma$ are compatible with the identifications made in creating $X_\Sigma$, so that $X_\Sigma$ contains the torus $(\mathbb{C}^*)^n$ as a Zariski open set. Furthermore, one can show that $X_\Sigma$ is a normal toric variety and that *all* normal toric varieties arise in this way, i.e., every normal toric variety is determined by a fan.

The toric variety $X_\Sigma$ is an example of an *abstract variety.* In particular, it can happen that $X_\Sigma$ is neither affine nor projective.

Here are some examples of toric varieties.

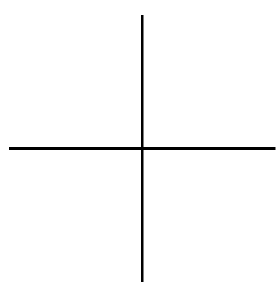
EXAMPLE 8.2. Given $\sigma \subset \mathbb{R}^n$, we get a fan by taking all faces of $\sigma$ (including $\sigma$ itself). The toric variety of this fan is the affine toric variety $U_\sigma$. $\square$

EXAMPLE 8.3. The fan for $\mathbb{P}^1$ is as follows:

The cones $\sigma_1 = [0, \infty)$ and $\sigma_2 = (-\infty, 0]$ give $U_{\sigma_1}$ with coordinate ring $\mathbb{C}[t]$ and $U_{\sigma_2}$ with coordinate ring $\mathbb{C}[t^{-1}]$, which patch in the usual way to give $\mathbb{P}^1$. $\square$

EXAMPLE 8.4. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$, and set $e_0 = -e_1 - \cdots - e_n$. Then we get a fan by taking the cones generated by all proper subsets of $\{e_0, e_1, \ldots, e_n\}$. You should check that the associated toric variety is $\mathbb{P}^n$. When $n = 2$, this gives the fan (3.2). $\square$

EXAMPLE 8.5. The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ is as follows:



In this figure, the 1-dimensional cones are four rays emanating from the origin and the 2-dimensional cones are the four quadrants. Thus the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has four 2-dimensional cones $\sigma_1, \ldots, \sigma_4$. The affine toric varieties $U_{\sigma_i} \simeq \mathbb{C}^2$ glue together in the usual way to give $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

There are *many* other nice examples of toric varieties. Later in this article we will see that every lattice polytope in $\mathbb{R}^n$ determines a projective toric variety.

## 9. Properties of Toric Varieties

The fan $\Sigma$ has a close relation to the structure of toric variety $X_\Sigma$. The basic idea is that there are one-to-one correspondences between the following objects:

- The limits $\lim_{t\to 0}\lambda^u(t)$ for $u \in |\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma$ ($|\Sigma|$ is the *support* of $\Sigma$).
- The cones $\sigma \in \Sigma$.
- The orbits of the torus action on $X_\Sigma$.

The correspondences is as follows: an orbit corresponds to a cone $\sigma$ if and only if $\lim_{t\to 0}\lambda^u(t)$ exists and lies in the orbit for all $u$ in the relative interior of $\sigma$. For an orbit $\mathrm{orb}(\sigma)$, we have:

- $\dim\sigma + \dim \mathrm{orb}(\sigma) = n$.
- $\mathrm{orb}(\sigma) \subset \overline{\mathrm{orb}(\tau)}$ if and only if $\tau \subset \sigma$.

In particular, the fixed points of the torus action correspond to the $n$-dimensional cones in the fan. It is a good exercise verify all of this for $\mathbb{P}^2$ and the fan drawn in Example 8.5.

We next discuss some basic properties of toric varieties. First, we need some terminology:

- A cone is *smooth* if it is generated by a subset of a basis of $\mathbb{Z}^n$.
- A cone is *simplicial* if it is generated by a subset of a basis of $\mathbb{R}^n$.

Then we have the following result.

THEOREM 9.1. *Let $X_\Sigma$ be the toric variety determined by a fan $\Sigma$ in $\mathbb{R}^n$. Then:*

(1) *$X_\Sigma$ is compact $\iff$ its support $|\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma$ is all of $\mathbb{R}^n$.*
(2) *$X_\Sigma$ is smooth $\iff$ every $\sigma\in\Sigma$ is smooth.*
(3) *$X_\Sigma$ has at worst finite quotient singularities $\iff$ every $\Sigma$ is simplicial. (Such toric varieties are called **simplicial**.)*

Since 2-dimensional cones are simplicial, toric surfaces have at worst finite quotient singularities. Furthermore, the finitely many singular points correspond to 2-dimensional cones whose minimal generators do not span $\mathbb{Z}^2$ over $\mathbb{Z}$.

## 10. Homogeneous Coordinates

We next describe homogeneous coordinates for toric varieties. Homogeneous coordinates on $\mathbb{P}^n$ give not only the graded ring $\mathbb{C}[x_0,\ldots,x_n]$ but also the quotient construction $\mathbb{P}^n \simeq (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$. Given a toric variety $X_\Sigma$, we generalize this as follows. Let $\rho_1,\ldots,\rho_r$ be the 1-dimensional cones of $\Sigma$ and let $\mathbf{n}_i \in \mathbb{Z}^n$ denote the primitive element of $\rho_i$ (= generator of $\rho_i \cap \mathbb{Z}^n$). Then introduce variables $x_i$ for $i = 1,\ldots,r$. The goal is to represent $X_\Sigma$ as the quotient

$$X_\Sigma = (\mathbb{C}^r \setminus Z)/G \tag{10.1}$$

for some variety $Z \subset \mathbb{C}^r$ and some group $G \subset (\mathbb{C}^*)^r$.

We define $Z$ as follows. For each cone $\sigma\in\Sigma$, we get the monomial

$$x^{\hat\sigma} = \textstyle\prod_{\mathbf{n}_i\notin\sigma} x_i$$

which is the product of all variables *not* coming from edges of $\sigma$. Then define

$$Z = \mathbf{V}(x^{\hat\sigma} \mid \sigma\in\Sigma) \subset \mathbb{C}^r.$$

In fact, $Z$ can be defined using only those $x^{\hat\sigma}$ which correspond to *maximal* cones of $\Sigma$ (= those cones not contained in any larger cone).

EXAMPLE 10.1. For $\mathbb{P}^n$, the $\mathrm{n}_i$ consist of the standard basis $e_1, \ldots, e_n$ plus $e_0 = -\sum_{i=1}^n$. This gives variables $x_0, \ldots, x_n$. Furthermore, the maximal cones of the fan are generated by the $n$-element subsets of $\{e_0, \ldots, e_n\}$. It follows that

$$Z = \mathbf{V}(x_0, \ldots, x_n) = \{(0, \ldots, 0)\} \subset \mathbb{C}^{n+1}.$$

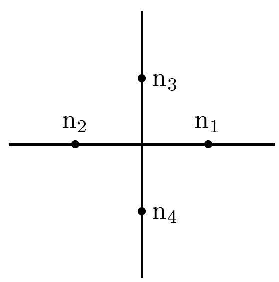
This of course is what we want for $\mathbb{P}^n$. □

There is another description of $Z$ due to Batyrev which is useful in practice. We say that a set of edge generators $\{\mathbf{n}_{i_1}, \ldots, \mathbf{n}_{i_s}\}$ is *primitive* if they don't lie in any cone of $\Sigma$ but every proper subset does. Then one can show that

$$Z = \bigcup_{\{\mathbf{n}_{i_1}, \ldots, \mathbf{n}_{i_s}\} \text{ primitive}} \mathbf{V}(x_{i_1}, \ldots, x_{i_s}).$$

This shows that $Z$ is a union of coordinate subspaces.

EXAMPLE 10.2. Consider the fan for $\mathbb{P}^1 \times \mathbb{P}^1$, where we have indicated the minimal generators $\mathbf{n}_1 = e_1, \mathbf{n}_2 = -e_1, \mathbf{n}_3 = e_2, \mathbf{n}_4 = -e_2$.




The only primitive sets are $\{\mathbf{n}_1, \mathbf{n}_2\}$ and $\{\mathbf{n}_3, \mathbf{n}_4\}$. It follows that

$$Z = \mathbf{V}(x_1, x_2) \cup \mathbf{V}(x_3, x_4) = \big(\{(0,0)\} \times \mathbb{C}^2\big) \cup \big(\mathbb{C}^2 \times \{(0,0)\}\big) \subset \mathbb{C}^4.$$

This will be useful shortly. □

We next describe the group $G$. This is the subgroup of $(\mathbb{C}^*)^r$ defined by

$$G = \{(\mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^r \mid \textstyle\prod_{i=1}^r \mu_i^{\langle \mathbf{m}, \mathbf{n}_i \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^n\}.$$

However, it suffices to let $\mathbf{m}$ be the standard basis elements $e_1, \ldots, e_n$. Thus $(\mu_1, \ldots, \mu_n) \in G$ if and only if

$$\textstyle\prod_{i=1}^r \mu_i^{\langle e_1, \mathbf{n}_i \rangle} = \prod_{i=1}^r \mu_i^{\langle e_2, \mathbf{n}_i \rangle} = \cdots = \prod_{i=1}^r \mu_i^{\langle e_n, \mathbf{n}_i \rangle} = 1. \tag{10.2}$$

Here are some examples.

EXAMPLE 10.3. For $\mathbb{P}^n$, Example 10.1 showed that the $\mathbf{n}_i$ are given by $e_0 = -\sum_{i=1}^n e_i$, $e_1, \ldots, e_n$. By (10.2), it follows that $(\mu_0, \ldots, \mu_n) \in G$ if and only if

$$\mu_0^{-1}\mu_1 = \mu_0^{-1}\mu_2 = \cdots = \mu_0^{-1}\mu_n = 1.$$

Thus $G = \{(\mu, \ldots, \mu) \in (\mathbb{C}^*)^{n+1}\} \simeq \mathbb{C}^*$. This gives the usual action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$. Since we know $Z$ from Example 10.1, the quotient representation (10.1) becomes

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

which is the usual way of expressing $\mathbb{P}^n$ as a quotient. □

EXAMPLE 10.4. For $\mathbb{P}^1 \times \mathbb{P}^1$, Example 10.2 showed that $\mathbf{n}_1 = e_1, \mathbf{n}_2 = -e_1, \mathbf{n}_3 = e_2, \mathbf{n}_4 = -e_2$. By (10.2), it follows that $(\mu_1, \mu_2, \mu_3, \mu_4) \in G$ if and only if

$$\mu_1 \mu_2^{-1} = \mu_3 \mu_4^{-1} = 1.$$

Hence $G = \{(\mu, \mu, \lambda, \lambda) \in (\mathbb{C}^*)^4\} \simeq (\mathbb{C}^*)^2$. Since we know $Z$ from Example 10.2, the quotient representation (10.1) becomes

$$\mathbb{P}^1 \times \mathbb{P}^1 = \Big(\mathbb{C}^4 \setminus \big(\{(0,0)\} \times \mathbb{C}^2\big) \cup \big(\mathbb{C}^2 \times \{(0,0)\}\big)\Big)/(\mathbb{C}^*)^2. \tag{10.3}$$

This might look complicated, but since $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{C}^*$, we have

$$\mathbb{P}^1 \times \mathbb{P}^1 = \Big((\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{C}^*\Big) \times \Big((\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{C}^*\Big).$$

This easily reduces to the quotient (10.3). □

Here is a precise statement of the quotient representation (10.1).

THEOREM 10.5. *If $X_\Sigma$ is a toric variety where $\mathbf{n}_1, \dots, \mathbf{n}_r$ span $\mathbb{R}^n$, then:*

(1) *$X_\Sigma$ is the universal categorical quotient $(\mathbb{C}^r \setminus Z)/G$.*
(2) *$X_\Sigma$ is a geometric quotient $(\mathbb{C}^r \setminus Z)/G$ if and only if $X_\Sigma$ is simplicial.*

This result was discovered independently by several people in the early 1990s (see [20]). Also note that the theorem uses the terms "universal categorical quotient" and "geometric quotient". The latter is the algebro-geometric analog of the usual idea of the quotient under a group action. As we will see in the next section, universal categorical quotient are not as well-behaved. Toric surfaces are always simplicial, so that $(\mathbb{C}^r \setminus Z)/G$ is always a geometric quotient in this case.

While we won't prove Theorem 10.5, we should at least explain why the quotient $(\mathbb{C}^r \setminus Z)/G$ contains the torus $(\mathbb{C}^*)^n$. For this, consider the map

$$(\mathbb{C}^*)^r \longrightarrow (\mathbb{C}^*)^n \tag{10.4}$$

which sends $(\mu_1, \dots, \mu_r)$ to $(t_1, \dots, t_n)$, where

$$t_j = \prod_{i=1}^r \mu_i^{\langle e_j, \mathbf{n}_i \rangle}, \ e_j = j\text{th standard basis vector.} \tag{10.5}$$

Then one can show that (10.4) is onto when $\mathbf{n}_1, \dots, \mathbf{n}_r$ span $\mathbb{R}^n$ as in Theorem 10.5. Furthemore, $(\mu_1, \dots, \mu_r)$ is in the kernel of (10.4) precisely when $t_j = 1$ for all $j$. Comparing (10.5) and (10.2), it follows that the kernel is the group $G$. Thus we have an isomorphism

$$(\mathbb{C}^*)^n \simeq (\mathbb{C}^*)^r/G,$$

so that the inclusion $(\mathbb{C}^*)^r \subset \mathbb{C}^r \setminus Z$ induces

$$(\mathbb{C}^*)^n \simeq (\mathbb{C}^*)^r/G \subset (\mathbb{C}^r \setminus Z)/G = X_\Sigma.$$

This explains why the quotient contains $(\mathbb{C}^*)^n$. Furthermore, since the "big" torus $(\mathbb{C}^*)^r$ acts naturally on $\mathbb{C}^r \setminus Z$, it follows that $(\mathbb{C}^*)^n$ acts on the quotient. Thus $(\mathbb{C}^r \setminus Z)/G$ is a toric variety, and it is also normal since categorical quotients preserve normality. In fact, one can *define* $X_\Sigma$ to be the quotient $(\mathbb{C}^r \setminus Z)/G$.

We conclude this section with a discussion of the polynomial ring

$$S = \mathbb{C}[x_1, \dots, x_r].$$

The key observation is that the action of $G$ induces a natural grading on this ring. If $f = f(x_1, \dots, x_r) \in S$ and $(\mu_1, \dots, \mu_r) \in G$, then $(\mu_1, \dots, \mu_r)$ acts on $f$ via

$$(\mu_1, \dots, \mu_r) \cdot f = f(\mu_1 x_1, \dots, \mu_r x_r).$$

As we will see, this induces a grading on $S$.

Example 10.6. For $\mathbb{P}^n$, the action of $G$ on a monomial is given by

$$(\mu, \dots, \mu) \cdot x_0^{a_0} \cdots x_n^{a_n} = (\mu x_0)^{a_0} \cdots (\mu x_n)^{a_n} = \mu^{a_0 + \cdots + a_n} x_0^{a_0} \cdots x_n^{a_n}.$$

We say that $x_0^{a_0} \cdots x_n^{a_n}$ has degree $a_0 + \cdots + a_n$, so that in particular, the $x_i$ all have degree 1. This is the usual grading on $S = \mathbb{C}[x_0, \dots, x_n]$. □

Example 10.7. For $\mathbb{P}^1 \times \mathbb{P}^1$, the action of $G$ on a monomial is given by

$$(\mu, \mu, \lambda, \lambda) \cdot x_1^a x_2^b x_3^c x_4^d = (\mu x_1)^a (\mu x_2)^b (\lambda x_3)^c (\lambda x_4)^d = \mu^{a+b} \lambda^{c+d} x_1^a x_2^b x_3^c x_4^d.$$

We say that $x_1^a x_2^b x_3^c x_4^d$ has degree $(a+b, c+d)$, so that in particular, $x_1, x_2$ have degree $(1,0)$ and $x_3, x_4$ have degree $(0,1)$. This is the usual bigrading on $S = \mathbb{C}[x_1, x_2, x_3, x_4]$. □

In general, $S = \mathbb{C}[x_1, \dots, x_r]$ has a grading so that two monomials have the same degree if and only if $G$ acts on them in the same way. One can prove that

$$\begin{aligned} &\deg(x_1^{a_1} \cdots x_r^{a_r}) = \deg(x_1^{b_1} \cdots x_r^{b_r}) \iff \\ &\text{there is } \mathbf{m} \in \mathbb{Z}^n \text{ such that } a_i = b_i + \langle \mathbf{n}_i, \mathbf{m} \rangle \text{ for all } i. \end{aligned}$$

With this grading, we call $S = \mathbb{C}[x_1, \dots, x_r]$ the *homogeneous coordinate ring* of $X_\Sigma$, and $f \in S$ is *homogeneous* if all monomials appearing in $f$ have the same degree in the above sense.

We will give some surprising examples of degrees in the next section.

## 11. Examples of Homogeneous Coordinates

Our first example shows that variables can have negative degree.

Example 11.1. We will construct global coordinates for the blow-up of $0 \in \mathbb{C}^n$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Z}^n$ and let $\sigma = \mathbb{R}^n_{\geq 0}$ be the cone they generate. The resulting affine toric variety is $\mathbb{C}^n$. Then set $e_0 = e_1 + \cdots + e_n$ and consider the fan $\Sigma$ whose cones are generated by all proper subsets of $\{e_0, \dots, e_n\}$, excluding $\{e_1, \dots, e_n\}$. We will prove that $X_\Sigma$ is the blow-up of $0 \in \mathbb{C}^n$ using the quotient representation (10.1). Let $x_i$ be the variable corresponding to the edge generated by $e_i$ for $i = 0, \dots, n$.

We begin with $Z$. The only primitive set is $\{e_1, \dots, e_n\}$, so that

$$Z = \mathbf{V}(x_1, \dots, x_n) = \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}.$$

As for $G$, the methods of the previous section show that $(\mu_0, \dots, \mu_n) \in G$ if and only if

$$\mu_0 \mu_1 = \mu_0 \mu_2 = \cdots = \mu_0 \mu_n = 1.$$

since $e_0 = e_1 + \cdots + e_n$. Hence $G = \{(\mu^{-1}, \mu, \dots, \mu) \in (\mathbb{C}^*)^{n+1}\} \simeq \mathbb{C}^*$, so that $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n$ by

$$\mu \cdot (x_0, \mathbf{x}) = (\mu^{-1} x_0, \mu \mathbf{x}).$$

It follows that the homogeneous coordinate ring of $X_\Sigma$ is $\mathbb{C}[x_0,\dots,x_n]$ where $\deg(x_0) = -1$ and $\deg(x_i) = +1$ for $1 \le i \le n$. Furthermore, we get the quotient representation

$$X_\Sigma = \big((\mathbb{C}\times\mathbb{C}^n)\setminus(\mathbb{C}\times\{0\})\big)/\mathbb{C}^* = \big(\mathbb{C}\times(\mathbb{C}^n\setminus\{0\})\big)/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts as above.

To analyze this quotient, take $(x_0,\mathbf{x}) \in \mathbb{C}\times(\mathbb{C}^n\setminus\{0\}$. We can act on this point using $G$ to obtain

$$\begin{array}{ll} x_0^{-1}\cdot(x_0,\mathbf{x}) = (1,x_0\mathbf{x}) & \text{if } x_0 \neq 0 \\ \mu\cdot(0,\mathbf{x}) = (0,\mu\mathbf{x}) & \text{if } \mu \neq 0. \end{array}$$

In the first line, $x_0\mathbf{x} \neq 0$, so the part of the quotient where $x_0 \neq 0$ is clearly $\mathbb{C}^n\setminus\{0\}$. In the second line, we see that the part of the quotient where $x_0 = 0$ is $\mathbb{P}^{n-1}$. Note also that the map $X_\Sigma \to \mathbb{C}^n$ given by $(x_0,x_1,\dots,x_n) \mapsto (x_0x_1,\dots,x_0x_n)$ is well-defined since $x_0x_i$ has degree 0 and hence is invariant under the group action. It follows that $X_\Sigma$ is the blow-up of $0 \in \mathbb{C}^n$. □

If $\sigma$ is an $n$-dimensional cone in $\mathbb{R}^n$, then the representation of $U_\sigma$ given by (10.1) is of the form $\mathbb{C}^r/G$, where $r$ is the number of edges of $\sigma$. This follows because $Z = \emptyset$ since a single cone has no primitive sets. Furthermore, there are two cases where $G$ can be determined explicitly:

- For $\sigma$ smooth, $G = \{1\}$, so that (10.1) gives $U_\sigma = \mathbb{C}^n$.
- For $\sigma$ simplicial, $G \simeq \mathbb{Z}^n/(\mathbb{Z}\mathbf{n}_1 + \cdots + \mathbb{Z}\mathbf{n}_\ell)$, so that according to (10.1), $U_\sigma = \mathbb{C}^n/G$ is the quotient of $\mathbb{C}^n$ by the finite group $G$.

Here is an example of the second bullet.

Example 11.2. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $\mathbf{n}_1 = (1,0)$ and $\mathbf{n}_2 = (1,2)$. What is the toric variety $U_\sigma$? As above, we know that $Z = \emptyset$, and by (10.2), $(\mu_1,\mu_2) \in G$ if and only if

$$\mu_1\mu_2 = \mu_2^2 = 1.$$

Thus $G = \{\pm(1,1)\} \subset (\mathbb{C}^*)^2$, so that $G \simeq \{\pm1\}$ acts on a monomial via

$$\pm1\cdot x_1^a x_2^b = (\pm1x_1)^a(\pm1x_2)^b = (\pm1)^{a+b}x_1^a x_2^b.$$

It follows that the homogeneous coordinate ring is $\mathbb{C}[x_1,x_2]$, where $x_1,x_2$ have degree $1 \bmod 2$. Thus $x_1^2, x_1x_2, x_2^2$ have degree $0 \bmod 2$. Furthermore, one can show the following:

1. $U_\sigma = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$.
2. $G$ acts on $\mathbb{C}^2$ by multiplication by $\pm1$.
3. The ring of invariants is $\mathbb{C}[x_1,x_2]^G = \mathbb{C}[x_1^2, x_1x_2, x_2^2]$.
4. The quotient map $\pi : \mathbb{C}^2 \to U_\sigma$ is given by $(x_1,x_2) \to (x_1^2, x_1x_2, x_2^2)$.

Note that $\mathbb{C}^2 \to U_\sigma$ is 2-to-1. This is a classic example of a finite quotient singularity. □

In the nonsimplicial case, things can be more complicated.

Example 11.3. Consider the cone $\sigma$ of Example 4.2. By Example 5.2, we know that $U_\sigma = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. The edge generators (4.1) of $\sigma$ give variables $x_1,x_2,x_3,x_4$. We leave it as an exercise for the reader to verify the following:

1. $G = \mathbb{C}^*$ acts on $\mathbb{C}^4$ via $\lambda\cdot(x_1,x_2,x_3,x_4) = (\lambda x_1, \lambda^{-1}x_2, \lambda^{-1}x_3, \lambda x_4)$.

2. In the homogeneous coordinate ring $\mathbb{C}[x_1, x_2, x_3, x_4]$, the variables have degrees
$$\deg(x_1) = \deg(x_4) = 1,\ \deg(x_2) = \deg(x_3) = -1.$$
3. The ring of invariants is
$$\mathbb{C}[x_1, x_2, x_3, x_4]^G = \mathbb{C}[x_1x_2, x_3x_4, x_1x_3, x_2x_4].$$
4. The quotient map $\pi : \mathbb{C}^4 \to \mathbb{C}^4/G = U_\sigma$ is
$$\pi(x_1, x_2, x_3, x_4) = (x_1x_2, x_3x_4, x_1x_3, x_2x_4).$$

In this example, the quotient $\mathbb{C}^4/G$ is a categorical quotient. To see how this can differ from an ordinary quotient, let $p \in U_\sigma$. Then one can show the following:

- $p \neq (0,0,0,0) \Rightarrow \pi^{-1}(p)$ is a $G$-orbit.
- $p = (0,0,0,0) \Rightarrow \pi^{-1}(p) = (\mathbb{C} \times \{0\} \times \{0\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C} \times \mathbb{C} \times \{0\})$.

The first bullet shows that most of the time, the categorical quotient $\mathbb{C}^4/G$ behaves like an ordinary quotient. However, things mess up over $(0,0,0,0)$ since the second bullet shows that the stuff mapping to $(0,0,0,0)$ has dimension 2 and hence consists of infinitely many $G$-orbits. $\square$

In general, quotients are not easy to construct in algebraic geometry. The above example gives one way of constructing a categorical quotient via the ring of invariants (item 3 above) under the group action. The idea is that the ring of invariants gives the coordinate ring of the quotient, and then one constructs the variety from its coordinate ring.

## 12. The Toric Variety of a Polytope

A *lattice polytope* $\Delta$ in $\mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{Z}^n$. We will show that an $n$-dimensional lattice polytope $\Delta$ determines a projective toric variety $X_\Delta$ of dimension $n$.

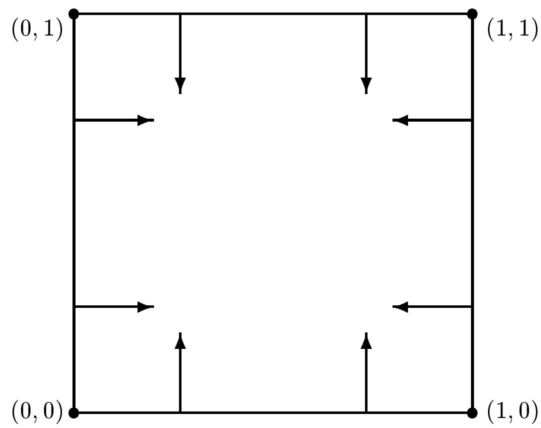
To do this, we first represent $\Delta$ as an intersection of halfspaces. For each facet $F$ of $\Delta$, there is an inward normal primitive vector $\mathbf{n}_F \in \mathbb{Z}^n$ and integer $a_F$ such that

$$\Delta = \bigcap_{F \text{ is a facet}} \{\mathbf{m} \in \mathbb{R}^n \mid \langle \mathbf{m}, \mathbf{n}_F \rangle \geq -a_F\}. \tag{12.1}$$

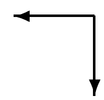
Given any face $\mathcal{F}$ of $\Delta$, let $\sigma_{\mathcal{F}}$ be the cone generated by $\mathbf{n}_F$ for all facets $F$ containing $\mathcal{F}$. Then
$$\Sigma_\Delta = \{\sigma_{\mathcal{F}} \mid \mathcal{F} \text{ is a face of } \Delta\}$$
is a fan which is called the *normal fan* of $\Delta$. This gives a toric variety denoted $X_\Delta$.

EXAMPLE 12.1. The unit square $\square$ with vertices $(0,0), (1,0), (1,1), (0,1)$ can be represented
$$\begin{aligned}\square &= \{a \geq 0\} \cap \{a \leq 1\} \cap \{b \geq 0\} \cap \{b \leq 1\}\\ &= \{a \geq 0\} \cap \{-a \geq -1\} \cap \{b \geq 0\} \cap \{-b \geq -1\}.\end{aligned}$$

It follows that the inward normals are $\pm e_1$ and $\pm e_2$ in $\mathbb{Z}^2$. These can be pictured as follows (not drawn to scale):




Each inward normal appears twice to show that each vertex gives a 2-dimensional cone in the normal fan. For example, the vertex $(1,1)$ gives the 2-dimensional cone



The other vertices are handled similarly, and the resulting normal fan is the one appearing in Example 8.5. Hence $X_\square = \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

In general, we can characterize these fans as follows.

THEOREM 12.2. *The normal toric variety of a fan $\Sigma$ in $\mathbb{R}^n$ is projective if and only if $\Sigma$ is the normal fan of an $n$-dimensional lattice polytope in $\mathbb{R}^n$.*

This is proved as follows. Let $\mathbf{m}_1, \ldots, \mathbf{m}_\ell$ be the lattice points of $\Delta$, so that $\ell = |\Delta \cap \mathbb{Z}^n|$. In the next section we will show that the map

$$\varphi(t_1, \ldots, t_n) = \big(\mathbf{t}^{\mathbf{m}_1}(t_1, \ldots, t_n), \ldots, \mathbf{t}^{\mathbf{m}_\ell}(t_1, \ldots, t_n)\big) \in \mathbb{P}^{\ell-1} \tag{12.2}$$

from $(\mathbb{C}^*)^n$ to $\mathbb{P}^{\ell-1}$ extends to a map $X_\Delta \to \mathbb{P}^{\ell-1}$. Notice that this is a projective version of (5.2). Then Theorem 12.2 is proved by showing that for $\nu \gg 0$, the corresponding map for $\nu\Delta$ is an embedding.

An important consequence of the previous paragraph is that it gives a completely elementary way to define the toric variety $X_\Delta$. Namely, given $\Delta$ and $\nu > 0$, the analog of (12.2) is the map

$$\varphi_\nu : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^{\ell_\nu - 1}$$

defined by the lattice points of $\nu\Delta$ (so that $\ell_\nu = |\nu\Delta \cap \mathbb{Z}^n|$). Then, provided $\nu$ is sufficiently large (we will explain how large in the next section), one can *define* $X_\Delta$ to be the Zariski closure of the image of $\varphi_\nu$. Notice how this is analogous to the definition of the affine toric variety $U_\sigma$ given in Section 5.

A useful observation is that the polytope $\Delta$ is combinatorially dual to its normal fan $\Sigma_\Delta$. This means that there is a one-to-one inclusion reversing correspondence

$$\sigma_{\mathcal{F}} \in \Sigma_\Delta \longleftrightarrow \mathcal{F} \subset \Delta$$

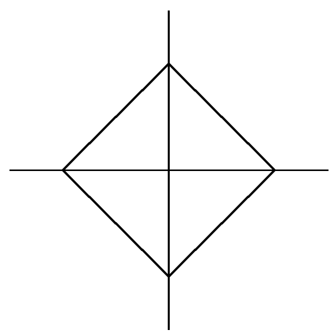
between cones of $\Sigma_\Delta$ and faces of $\Delta$ (provided we count $\Delta$ as a face of itself) such that

$$\dim \sigma_{\mathcal{F}} + \dim\, \mathcal{F} = n \tag{12.3}$$

for all faces $\mathcal{F}$ of $\Delta$. Combining this with the correspondence between cones in $\Sigma_\Delta$ and torus orbits in $X_\Delta$ from Section 9, we get a one-to-one dimension preserving correspondence between faces of $\Delta$ and torus orbits of $X_\Delta$. Thus $\Delta$ determines the combinatorics of the toric variety $X_\Delta$.

There is also a dual construction of $X_\Delta$. Suppose that $P \subset \mathbb{R}^n$ is an $n$-dimensional polytope which contains the origin as an interior point and whose vertices lie in $\mathbb{Q}^n$. Then we get a fan $\Sigma_P$ in $\mathbb{R}^n$ by taking cones (relative to the origin) over the faces of $P$. The resulting toric variety is denoted $X_P$.

EXAMPLE 12.3. Consider the tilted square $P$ in the plane:



The fan $\Sigma_P$ obtained by taking cones over faces is the fan of Example 8.5. It follows immediately that $X_P = \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

To relate this to our earlier construction, we define the *polar* or *dual* of $P \subset \mathbb{R}^n$ to be

$$P^\circ = \{\mathbf{m} \in \mathbb{R}^n \mid \langle \mathbf{m}, \mathbf{u}\rangle \geq -1 \text{ for all } \mathbf{u} \in P\}.$$

Since $P$ has rational vertices, so does $P^\circ$, which means that $\Delta = \ell P^\circ$ is a lattice polytope for some positive integer $\ell$. Then one can show that $\Sigma_P$ is the normal fan of $\Delta$, so that $X_P$ is the projective toric variety $X_\Delta$.

## 13. Polytopes and Homogeneous Coordinates

As in the previous section, we fix a lattice polytope $\Delta \subset \mathbb{R}^n$. The homogeneous coordinates of $X_\Delta$ have a nice description as follows. By (12.3), 1-dimensional cones of the normal fan correspond to facets of $\Delta$. It follows that variables correspond to facets. If we label the facets $F_1, \ldots, F_r$ and the inner normals $\mathbf{n}_1, \ldots, \mathbf{n}_r$, then we call $x_1, \ldots, x_r$ the *facet variables* of the polytope $\Delta$.

Also, the exceptional set $Z = \mathbf{V}(x^{\hat\sigma} \mid \sigma \in \Sigma_\Delta, \dim(\sigma) = n) \subset \mathbb{C}^r$ has a nice description. By (12.3), $n$-dimension cones in the normal fan correspond to vertices $v \in \Delta$. Thus we set $x^{\hat v} = x^{\hat\sigma}$. We call this the *vertex monomial of* $v$ since it is the product of those variables whose facets miss the vertex $v$. It follows that $Z$ is defined by the vanishing of the vertex monomials, so that $\mathbb{C}^r \setminus Z$ consists of points in $\mathbb{C}^r$ where at least one vertex monomial is nonvanishing.

From $\Delta$, we get some interesting monomials in the homogeneous coordinate ring $\mathbb{C}[x_1,\ldots,x_r]$. Write (12.1) as

$$\Delta = \bigcap_i \{\mathbf{m} \in \mathbb{R}^n \mid \langle \mathbf{m}, \mathbf{n}_i \rangle \geq -a_i\}. \tag{13.1}$$

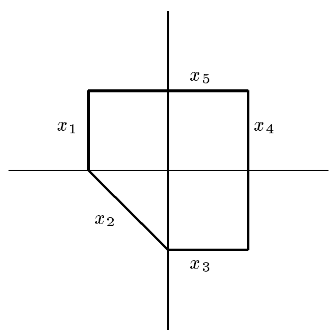
Then, given $\mathbf{m} \in \Delta \cap \mathbb{Z}^n$, set

$$\mathbf{x}^{\mathbf{m}} = \prod_{i=1}^r x_i^{\langle \mathbf{m}, \mathbf{n}_i\rangle + a_i}.$$

We call $\mathbf{x}^{\mathbf{m}}$ a $\Delta$-*monomial.* The description (13.1) of $\Delta$ shows that the exponents of $\mathbf{x}^{\mathbf{m}}$ are all $\geq 0$, so that $\mathbf{x}^{\mathbf{m}}$ is in the homogeneous coordinate ring.

One nice observation is that the exponent of $x_i$ in $\mathbf{x}^{\mathbf{m}}$ gives the *lattice distance* from $\mathbf{m}$ to the facet $F_i$. To see this, suppose that the exponent of $x_i$ is $a > 0$. Then $F_i$ lies in the hyperplane $\{m \in \mathbb{R}^n \mid \langle m, \mathbf{n}_i\rangle + a_i = 0\}$. To get from here to $\mathbf{m}$, we must pass through the $a$ parallel hyperplanes, namely $\{m \in \mathbb{R}^n \mid \langle m, \mathbf{n}_i\rangle + a_i = j\}$ for $j = 1,\ldots,a$. Here is an example.

Example 13.1. Consider the toric variety $X_\Delta$ of the polytope $\Delta \subset \mathbb{R}^2$




with vertices $(1,1),(-1,1),(-1,0),(0,-1),(1,-1)$. In terms of (13.1), we have $a_1 = \cdots = a_5 = 1$, where the indices correspond to the variables $x_1,\ldots,x_5$ shown in the above picture. The 8 points of $\Delta \cap \mathbb{Z}^2$ give the following $\Delta$-monomials:

$$\begin{array}{ccc} x_2x_3^2x_4^2, & x_1x_2^2x_3^2x_4, & x_1^2x_2^3x_3^2 \\ x_3x_4^2x_5, & x_1x_2x_3x_4x_5, & x_1^2x_2^2x_3x_5 \\ & x_1x_4x_5^2, & x_1^2x_2x_5^2. \end{array}$$

In this display, the position of each $\Delta$-monomial $\mathbf{x}^{\mathbf{m}}$ corresponds to the position of the lattice point $\mathbf{m} \in \Delta \cap \mathbb{Z}^2$. □

One nice property is all $\Delta$-monomials have the same degree. To see this, let $\mu = (\mu_1,\ldots,\mu_r) \in G$ and set

$$\mu_\Delta = \prod_i \mu_i^{a_i}.$$

Then, given $\mathbf{m} \in \Delta \cap \mathbb{Z}^n$, we have

$$\mu \cdot \mathbf{x}^{\mathbf{m}} = \prod_{i=1}^r (\mu_i x_i)^{\langle \mathbf{m}, \mathbf{n}_i\rangle + a_i} = \mu_\Delta \mathbf{x}^{\mathbf{m}} \tag{13.2}$$

since $\prod_{i=1}^r \mu_i^{\langle \mathbf{m}, \mathbf{n}_i\rangle} = 1$ by the definition of $G$ given in Section 10. It follows that all $\Delta$-monomials transform the same way under $G$, which means that they have the same degree. Furthermore, one can show the $\Delta$-monomials give *all* monomials of this degree.

Here are three further observations:

- The lattice points in the interior $\mathrm{int}(\Delta)$ of $\Delta$ correspond precisely to those $\Delta$-monomials which are divisible by $x_1 \cdots x_r$.
- If $\nu$ is a positive integer, then $\Delta$ and $\nu\Delta$ have the same normal fan and toric variety. Thus $X_\Delta = X_{\nu\Delta}$.
- In particular, $X_\Delta$ and $X_{\nu\Delta}$ have the same coordinate ring $\mathbb{C}[x_1, \ldots, x_r]$. Furthermore, in a sense that can be made precise, the $\nu\Delta$-monomials are the monomials whose degree is $\nu$ times the degree of the $\Delta$-monomials.

Lattice points in $\mathrm{int}(\Delta)$ and $\nu\Delta$ play an important role in the *Ehrhart polynomial* of the polytope $\Delta$.

We can also use $\Delta$-monomials to give a homogeneous version of the map (12.2) which uses the quotient representation $X_\Delta = (\mathbb{C}^r \setminus Z)/G$. As in the previous section, let $\mathbf{m}_i, i = 1, \ldots, \ell$ be the lattice points of $\Delta \cap \mathbb{Z}^n$. Then consider the map

$$\mathbf{x} = (x_1, \ldots, x_r) \longrightarrow (\mathbf{x}^{\mathbf{m}_1}, \ldots, \mathbf{x}^{\mathbf{m}_\ell}). \tag{13.3}$$

First observe that if $v = \mathbf{m}_j$ is a vertex of $\Delta$, then $\mathbf{x}^{\mathbf{m}_j}$ and the vertex monomial $x^{\hat{v}}$ defined in Section 12 involve exactly the same variables. This is because $v$ has zero lattice distance to all facets it lies in and positive lattice distance to all the others. It follows that since $Z$ is defined by the vanishing of the vertex monomials, the map (13.3) gives a well defined map

$$\phi : \mathbb{C}^r \setminus Z \longrightarrow \mathbb{P}^{\ell-1}.$$

Furthermore, given $\mathbf{x} \in \mathbb{C}^r \setminus Z$ and $\mu \in G$, (13.2) implies that

$$\phi(\mu \cdot \mathbf{x}) = \mu_\Delta p(\mathbf{x}).$$

Since we are mapping to projective space, $\phi$ induces a well-defined map

$$X_\Delta = \big(\mathbb{C}^r \setminus \mathbf{V}(B)\big)/G \longrightarrow \mathbb{P}^{\ell-1}. \tag{13.4}$$

The surprise is that if one restricts this map to $(\mathbb{C}^*)^n \subset X_\Delta$, then the result is *exactly* the map (12.2)

$$\varphi(t_1, \ldots, t_n) = \big(\mathbf{t}^{\mathbf{m}_1}(t_1, \ldots, t_n), \ldots, \mathbf{t}^{\mathbf{m}_\ell}(t_1, \ldots, t_n)\big) \in \mathbb{P}^{\ell-1}$$

defined by the Laurent monomials of lattice points of $\Delta \cap \mathbb{Z}^n$. To prove this, observe that by (10.5), the variables $t_1, \ldots, t_n$ on the torus $(\mathbb{C}^*)^n$ are related to the variables $x_1, \ldots, x_r$ on $\mathbb{C}^r$ via

$$t_j = \prod_{i=1}^r x_i^{\langle e_j, \mathbf{n}_i \rangle}.$$

Now let $\mathbf{m} = (m_1, \ldots, m_n) = \sum_{j=1}^n m_j e_j \in \mathbb{Z}^n$. Then one computes that

$$\begin{aligned} x_1^{a_1} \cdots x_r^{a_r}\, \mathbf{t}^{\mathbf{m}} &= \prod_{i=1}^r x_i^{a_i} \prod_{j=1}^n t_j^{m_j} = \prod_{i=1}^r x_i^{a_i} \prod_{j=1}^n \Big(\prod_{i=1}^r x_i^{\langle e_j, \mathbf{n}_i \rangle}\Big)^{m_j} \\ &= \prod_{i=1}^r x_i^{\langle \mathbf{m}, \mathbf{n}_i \rangle + a_i} = \mathbf{x}^{\mathbf{m}}. \end{aligned}$$

When restricted to a point in $(\mathbb{C}^*)^n$, it follows that as we vary $\mathbf{m} \in \Delta \cap \mathbb{Z}^n$, the monomials $\mathbf{t}^{\mathbf{m}}$ and $\mathbf{x}^{\mathbf{m}}$ differ by a multiplicative factor which doesn't depend on

$\mathbf{m}$. Hence $\varphi$ and (13.4) give the same map on $(\mathbb{C}^*)^n$. In particular, this proves the claim made in the previous section that $\varphi$ extends to all of $X_\Delta$.

Finally, we note that while (13.4) is not an embedding in general, it is known to be an embedding in the following two cases:

- $X_\Delta$ is smooth, or
- We replace $\Delta$ with $(n-1)\Delta$, $n = \dim(\Delta)$.

In particular, when $\Delta$ is a polygon, we have $(n-1)\Delta = (2-1)\Delta = \Delta$. Thus (13.4) is always an embedding when $X_\Delta$ is a toric surface. This is the case of greatest interest in geometric modeling.

In our final example, we note that the map $\phi$ has appeared in the geometric modeling literature.

EXAMPLE 13.2. In the paper [22] by Rimas Krasauskas, the map (13.3) appears as equation (18) in Definition 14. Krasauskas denotes the homogeneous coordinates by $u_1, \ldots, u_r$ and the points of $\Delta \cap \mathbb{Z}^n$ by $m_0, \ldots, m_N$. He writes the $\Delta$-monomials as $u^{h(m_i)}$ instead of $\mathbf{x}^{\mathbf{m}_i}$. □

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