Minicourse on Toric Varieties

University of Buenos Aires

July, 2001

David A. Cox

Amherst College

Lecture I: Toric Varieties and Their Constructions

Lecture II: Toric Varieties and Polytopes

Lecture III: Toric Varieties and Mirror Symmetry

Acknowledgements

Bibliography

Lecture I: Toric Varieties and Their Constructions

1 Varieties

We will work over the complex numbers $\mathbb{C}$. Basic examples of varieties are:

- Affine space $\mathbb{C}^n$ and *affine varieties*

$$V = \mathbf{V}(f_1,\dots,f_s) \subset \mathbb{C}^n$$

  defined by the polynomial equations $f_1 = \cdots = f_s = 0$.

- Projective space $\mathbb{P}^n$ and *projective varieties*

$$V = \mathbf{V}(F_1,\dots,F_s) \subset \mathbb{P}^n$$

  defined by the homogeneous equations $F_1 = \cdots = F_s = 0$.

Example 1.1 Let $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Then $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ is an affine variety since the map $(t_1,\dots,t_n) \mapsto (t_1,\dots,t_n,1/t_1\cdots t_n)$ gives a bijection

$$(\mathbb{C}^*)^n \simeq \mathbf{V}(x_1x_2\cdots x_{n+1}-1) \subset \mathbb{C}^{n+1}.$$

We call $\mathbb{C}^*$ the *n-dimensional complex torus*.

Also recall that given varieties V and W, we can form the *product variety* $V\times W$. Then a *morphism* $\varphi : V \to W$ is a function whose graph is a subvariety of $V\times W$.

2 Characters and 1-Parameter Subgroups

The torus $T = (\mathbb{C}^*)^n$ has:

- The *character group*

$$M = \{\chi : T \to \mathbb{C}^* \mid \chi \text{ is a morphism and a group homomorphism}\}.$$

- The group of 1-*parameter subgroups*

$$N = \{\lambda : \mathbb{C}^* \to T \mid \lambda \text{ is a morphism and a group homomorphism}\}.$$

Note that:

- $M \simeq \mathbb{Z}^n$ where $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$ gives

$$\chi^m(t_1,\dots,t_n) = t_1^{m_1}\cdots t_n^{m_n}.$$

- $N \simeq \mathbb{Z}^n$ where $u = (u_1,\dots,u_n) \in \mathbb{Z}^n$ gives

$$\lambda^u(t) = (t^{u_1},\dots,t^{u_n}).$$

Given $\chi \in M$ and $\lambda \in N$, the composition $\chi \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ is of the form $t \mapsto t^k$ for some $k \in \mathbb{Z}$. If we set $k = \langle \chi, \lambda \rangle$, then

$$\chi \circ \lambda(t) = t^{\langle \chi, \lambda \rangle}.$$

One can easily check that:

- The map $M \times N \to \mathbb{Z}$ given by $(\chi, \lambda) \mapsto \langle \chi, \lambda \rangle$ is a perfect pairing.
- Given $m = (m_1, \ldots, m_n)$ and $u = (u_1, \ldots, u_n)$, then

$$\langle \chi^m, \lambda^u \rangle = m_1 u_1 + \cdots + m_n u_n.$$

Furthermore:

- We will usually identify M with $\mathbb{Z}^n$ and write $m \in M$. However, when we think of m as a function on $T = (\mathbb{C}^*)^n$, we continue to write χ^m.
- Similarly, we identity N with $\mathbb{Z}^n$ and write $u \in N$, though we write λ^u when thinking of u as a 1-parameter subgroup.
- Finally, we will usually write $\langle m, u \rangle$ instead of $\langle \chi^m, \lambda^u \rangle$.

3 Toric Varieties

The torus $T = (\mathbb{C}^*)^n$ can be regarded as a Zariski open subset of a larger variety X in many ways:

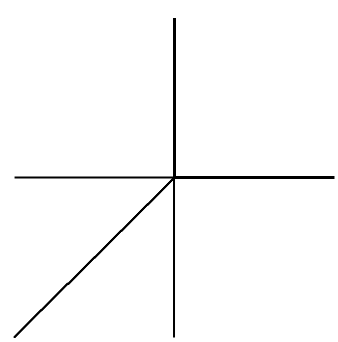
- $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ under the natural inclusion.
- $(\mathbb{C}^*)^n \subset \mathbb{P}^n$ under the map $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 1)$.
- $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$ contains the Zariski open set $V \cap (\mathbb{C}^*)^4$. The map $(r, s, t) \mapsto (r, s, t, rs/t)$ induces a bijection $(\mathbb{C}^*)^3 \simeq V \cap (\mathbb{C}^*)^4$. Thus V contains a copy of $(\mathbb{C}^*)^3$ as a Zariski open set.

Definition 3.1 *A* **toric variety** *is a normal variety X of dimension n which contains a torus $T = (\mathbb{C}^*)^n$ as a Zariski open set in such a way that the natural action of T on itself given by the group structure extends to an action of T on X.*

All of the above examples are toric varieties. The main goal of Lecture I is to explain three constructions of toric varieties. The common thread of these constructions is the rich combinatorial structure which underlies a toric variety. Here is an example,

Example 3.2 Let's show that $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$ gives the following picture in $\mathbb{R}^2 = N \otimes_{\mathbb{Z}} \mathbb{R}$:

(1.1)



A 1-parameter subgroup $u \in N$ gives a map $\lambda^u : \mathbb{C}^* \to \mathbb{P}^2$. $\mathbb{P}^2$ is complete, so that

$$\lim_{t\to 0} \lambda^u(t)$$

exists in $\mathbb{P}^2$. If $u = (a,b) \in \mathbb{Z}^2 = N$, then the description of λ^u given on page 1 implies that

$$\lambda^u(t) = (t^a, t^b, 1).$$

It is then straightforward to compute that

$$\lim_{t\to 0} \lambda^u(t) = \lim_{t\to 0}(t^a, t^b, 1) = \begin{cases} (0,0,1) & a,b>0 \\ (0,1,1) & a>0, b=0 \\ (1,0,1) & a=0, b>0 \\ (1,1,1) & a=b=0 \\ (0,1,0) & a>b, b<0 \\ (1,0,0) & a<0, a<b \\ (1,1,0) & a<0, a=b. \end{cases} \tag{1.2}$$

The first four cases are trivial. To see how the fifth case works, note that

$$\lim_{t\to 0}(t^a, t^b, 1) = \lim_{t\to 0}(t^{a-b}, 1, t^{-b})$$

since these are homogeneous coordinates. Then $a > b$ and $b < 0$ imply that the limit is $(0,1,0)$, as claimed. The last two cases are similar.

Now observe that (1.1) decomposes the plan into 7 disjoint regions:

- The open sets 1: $a,b > 0$, 2: $a = 0, b > 0$, 3: $a > b, b < 0$.
- The open rays 1: $a > 0, b = 0$, 2: $a = 0, b > 0$, 3: $a < 0, a = b$.
- The point $a = b = 0$.

The corresponds perfectly with (1.2). In the next section, we will see that (1.1) is the *fan* corresponding to the toric variety $\mathbb{P}^2$.

4 First Construction: Cones and Fans

Let X be a toric variety with $T = (\mathbb{C}^*)^n$, M and N as above. We first explain how the character group M leads to pictures generalizing (1.1). The idea is that $m \in M$ gives $\chi^m : T \to \mathbb{C}^*$. Since $T \subset X$, we can regard χ^m as a rational function on X.

The divisor of this rational function has some nice properties. It is supported on the complement of T in X. This complement will be a union of irreducible divisors, which we denote

$$X \setminus T = D_1 \cup \cdots \cup D_r.$$

Then the divisor of χ^m can be written

$$\mathrm{div}(\chi^m) = \sum_{i=1}^{r} a_i D_i,$$

where a_i is the order of vanishing (or the negative of the order of the pole) of χ^m along D_i. This is one of the reasons we require that X be normal—it ensures that the a_i are well-defined.

The key observation is that the map $m \mapsto a_i$ is a homormorphism. (**Exercise:** Prove this.) Using the duality between M and N, we get $n_i \in N$ such that $a_i = \langle m, n_i \rangle$. This implies

$$\operatorname{div}(\chi^m) = \sum_{i=1}^{r} \langle m, n_i \rangle D_i. \tag{1.3}$$

It follows that the toric structure of X uniquely determines a unique set of elements $n_1, \ldots, n_r \in N$. The n_i are part of the *fan* determined by X.

However, before we can define fans, we must consider *cones*. We will let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ denote the real vector spaces obtained from M and N.

A *rational polyhedral cone* $\sigma \subset N_{\mathbb{R}}$ is a cone generated by finitely many elements of N:

$$\sigma = \left\{ \lambda_1 u_1 + \cdots + \lambda_s u_s \in N_{\mathbb{R}} \mid \lambda_1, \ldots, \lambda_s \geq 0 \right\},$$

where $u_1, \ldots, u_s \in N$. Then:

- σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.
- The *dimension* of σ is the dimension of the smallest subspace containing σ.
- A *face* of σ is the intersection $\{\ell = 0\} \cap \sigma$, where ℓ is a linear form which is nonnegative on σ. The set of faces of σ of dimension r is denoted $\sigma(r)$.
- The *edges* of σ are the 1-dimensional faces $\rho \in \sigma(1)$. The *primitive element* n_ρ of $\rho \in \sigma(1)$ is the unique generator of $\rho \cap N$. The primitive elements $n_\rho,\ \rho \in \sigma(1)$, generate the σ.
- The *facets* of σ are the codimension-1 faces. When $\dim \sigma = n$, each facet has an inward pointing normal which is naturally an element of $M_{\mathbb{R}}$. We get a unique inward normal by requiring that it is in M and has minimal length.

If $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone, then its *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$ is

$$\sigma^\vee = \left\{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \right\}.$$

This is a rational polyhedral cone of dimension n. Then consider the semigroup algebra

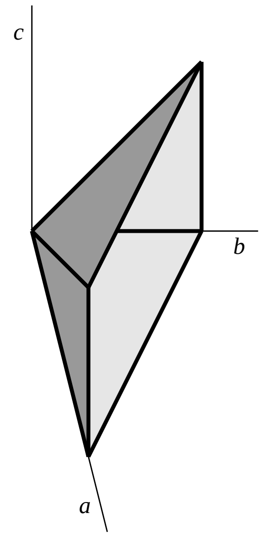
$$\mathbb{C}[\sigma^\vee \cap M]$$

consisting of linear combinations of characters χ^m, with multiplication given by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. *Gordan's Lemma* implies that $\mathbb{C}[\sigma^\vee \cap M]$ is a finitely generated algebra over $\mathbb{C}$.

Example 4.1 First consider an n-dimensional cone σ generated by a basis $e_1, \ldots, e_n$ of N. The basis gives an isomorphism $N \simeq \mathbb{Z}^n$ which takes σ to the "first quadrant" where all coordinates are nonnegative. In terms of the dual basis e_i^* of M, $\sigma^\vee$ has the same description. It follows that $\mathbb{C}[\sigma^\vee \cap M]$ can be identified with the usual polynomial ring $\mathbb{C}[t_1, \ldots, t_n]$ by setting $t_i = \chi^{e_i^*}$.

Example 4.2 Next suppose that $\sigma = \{0\}$ is the trivial cone. Then $\sigma^\vee = M_{\mathbb{R}}$, so that $\sigma^\vee \cap M = M$. Picking bases for N and M as in the previous example, the semigroup algebra $\mathbb{C}[M]$ can be identified with the ring of Laurent polynomials $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Example 4.3 Consider the cone $\sigma \subset \mathbb{R}^3$ pictured below:



The inward pointing normals of the facets of σ are

(1.4) $$m_1 = (1,0,0),\ m_2 = (0,1,0),\ m_3 = (0,0,1),\ m_4 = (1,1,-1).$$

These generate the dual cone $\sigma^\vee$ and in this case also generate the semigroup $\sigma^\vee \cap M$. Under the ring homomorphism $\mathbb{C}[x,y,z,w] \to \mathbb{C}[\sigma^\vee \cap M]$ defined by

$$x \mapsto \chi^{m_1},\ y \mapsto \chi^{m_2},\ z \mapsto \chi^{m_3},\ w \mapsto \chi^{m_4},$$

one sees that $xy - zw \mapsto 0$ since $m_1 + m_2 = m_3 + m_4$. It follows easily that

$$\mathbb{C}[x,y,z,w]/\langle xy - zw\rangle \simeq \mathbb{C}[\sigma^\vee \cap M].$$

In general, one can write $\mathbb{C}[\sigma^\vee \cap M] \simeq \mathbb{C}[x_1,\dots,x_N]/\langle f_1,\dots,f_s\rangle$, generalizing Example 4.3. Then the affine variety

$$X_\sigma = \mathbf{V}(f_1,\dots,f_s) \subset \mathbb{C}^N$$

is the *affine toric variety* determined by the strictly convex rational polyhedral cone σ. The construction of X_σ is a special case of the "Spec" of a ring, as described in Hartshorne. Thus

$$X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

Also, $\mathbb{C}[\sigma^\vee \cap M]$ is the *coordinate ring* of X_σ, which consists of all polynomial functions on X_σ. Note that the inclusion $\mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}[M]$ corresponds to an inclusion $T \subset X_\sigma$. Thus $\mathbb{C}[\sigma^\vee \cap M]$ tells us which characters on the torus T are allowed to extend to functions defined on all of X_σ.

You should check that the above examples give the following affine toric varieties:

- Example 4.1 gives $\mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[t_1,\dots,t_n])$.
- Example 4.2 gives $(\mathbb{C}^*)^n = \mathrm{Spec}(\mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}])$.
- Example 4.3 gives $V = \mathbf{V}(xy - zw) = \mathrm{Spec}(\mathbb{C}[x,y,z,w]/\langle xy - zw\rangle)$.

We next create more general toric varieties by gluing together affine toric varieties containing the same torus T. This brings us to the concept of a *fan*, which is defined to be a finite collection Σ of cones in $N_{\mathbb{R}}$ with the following three properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

Each $\sigma \in \Sigma$ gives an affine toric variety X_σ, and if τ is a face of σ, then X_τ can be regarded as a Zariski open subset of X_σ. This leads to the following definiton.

Definition 4.4 *Given a fan Σ in $N_{\mathbb{R}}$, X_Σ is the variety obtained from the affine varieties X_σ, $\sigma \in \Sigma$, by gluing together X_σ and X_τ along their common open subset $X_{\sigma\cap\tau}$ for all $\sigma, \tau \in \Sigma$.*

The inclusions $T \subset X_\sigma$ are compatible with the identifications made in creating X_Σ, so that X_Σ contains the torus T as a Zariski open set. Furthermore, one can show that X_Σ is a toric variety and that *all* toric varieties arise in this way, i.e., every toric variety is determined by a fan. Here are some examples.

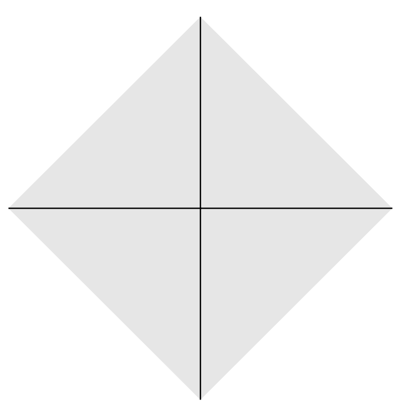
Example 4.5 Given $\sigma \subset N_{\mathbb{R}}$, we get a fan by taking all faces of σ (including σ). The toric variety of this fan is the affine toric variety X_σ. For the special case when σ is generated by the first k vectors of a basis $e_1, \ldots, e_n$ of N, you should check that

$$X_\sigma = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

Example 4.6 The fan for $\mathbb{P}^1$ is as follows:

The cones $\sigma_1 = [0, \infty)$ and $\sigma_2 = (-\infty, 0]$ give $X_1 = \mathrm{Spec}(\mathbb{C}[t]) \simeq \mathbb{C}$ and $X_2 = \mathrm{Spec}(\mathbb{C}[t^{-1}]) \simeq \mathbb{C}$, which patch in the usual way to give $\mathbb{P}^1$.

Example 4.7 The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ is as follows:



In this figure, 1-dimensional cones are indicated with thick lines, and 2-dimensional cones (which extend to infinity) are shaded. Thus the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has four 2-dimensional cones $\sigma_1, \dots, \sigma_4$. The affine toric varieties $X_{\sigma_i} \simeq \mathbb{C}^2$ glue together in the usual way to give $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 4.8 Let $e_1, \dots, e_n$ be a basis of $N = \mathbb{Z}^n$, and set $e_0 = -e_1 - \cdots - e_n$. Then we get a fan by taking the cones generated by all proper subsets of $\{e_0, e_1, \dots, e_n\}$. You should check that the associated toric variety is $\mathbb{P}^n$. When $n = 2$, this gives the fan (1.1).

There are *many* other nice examples of toric varieties, including products of projective spaces, weighted projective spaces, and Hirzebruch surfaces. We will see in the Lecture II that every lattice polytope in $M_\mathbb{R}$ determines a projective toric variety.

Toric varieties are sometimes called *torus embeddings*, and Fulton and Oda call the fan Δ. Also, the toric variety determined by Σ is variously denoted X_Σ, $X(\Sigma)$, $Z(\Sigma)$, and $T_N\mathrm{emb}(\Sigma)$. Furthermore, polytopes (which we will encounter in Lecture II) are denoted P, $\square$, and (just to confuse matters more) Δ. The lack of uniform notation is unfortunate, so that the reader of a paper using toric methods needs to look carefully at the notation.

5 Properties of Toric Varieties

The fan Σ has a close relation to the structure of toric variety X_Σ. The basic idea is that there are one-to-one correspondences between the following sets of objects:

- The limits $\lim_{t\to 0} \lambda^u(t)$ for $u \in |\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ ($|\Sigma|$ is the *support* of Σ).
- The cones $\sigma \in \Sigma$.
- The orbits O of the torus action T on X_Σ.

The correspondences is as follows: an orbit O corresponds to a cone σ if and only if $\lim_{t\to 0} \lambda^u(t)$ exists and lies in O for all u in the relative interior of σ. Setting $O = \mathrm{orb}(\sigma)$, we have

- $\dim \sigma + \dim \mathrm{orb}(\sigma) = n$.
- $\mathrm{orb}(\sigma) \subset \overline{\mathrm{orb}(\tau)}$ if and only if $\tau \subset \sigma$.

In particular, the fixed points of the torus action correspond to the n-dimensional cones in the fan. (**Exercise:** Verify all of this for $\mathbb{P}^2$ and the fan drawn in Example 3.2.)

We next discuss some basic properties of toric varieties. First, some terminology:

- A cone is *smooth* if it is generated by a subset of a basis of N.
- A cone is *simplicial* if it is generated by a subset of a basis of $N_\mathbb{R}$.

Then we have the following result.

Theorem 5.1 *Let X_Σ be the toric variety determined by a fan Σ in $N_\mathbb{R}$. Then:*

(a) *X_Σ is complete $\iff$ the support $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ is all of $N_\mathbb{R}$.*

(b) *X_Σ is smooth $\iff$ every $\sigma \in \Sigma$ is smooth.*

(c) *X_Σ is an orbifold $\iff$ every Σ is simplicial. (Such toric varieties are called* **simplicial**.)

(d) *X_Σ is Cohen-Macaulay with dualizing sheaf $\omega_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-\sum_\rho D_\rho)$.*

(e) *X_Σ has at worst rational singularities.*

Lecture II will give criteria for X_Σ to be projective.

6 Second Construction: Homogeneous Coordinates

Our second construction uses homogeneous coordinates for toric varieties. Homogeneous coordinates on $\mathbb{P}^n$ give not only the graded ring $\mathbb{C}[x_0,\ldots,x_n]$ but also the quotient construction $\mathbb{P}^n \simeq (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$. Given an arbitrary toric variety X_Σ, we generalize this as follows. For each $\rho \in \Sigma(1)$, introduce a variable x_ρ, which gives the polynomial ring

$$S = \mathbb{C}\big[x_\rho \mid \rho \in \Sigma(1)\big].$$

To grade S, we first note that for each $\rho \in \Sigma(1)$, the corresponding orbit has codimension 1, which means that its closure is an irreducible divisor $D_\rho \subset X_\Sigma$. It follows that a monomial $\prod_\rho x_\rho^{a_\rho}$ gives an effective divisor $D = \sum_\rho a_\rho D_\rho$. For this reason, we write the monomial as x^D. Now define the group $A_{n-1}(X_\Sigma)$ by the exact sequence

$$M \xrightarrow{\alpha} \bigoplus_\rho \mathbb{Z}D_\rho \xrightarrow{\beta} A_{n-1}(X_\Sigma) \longrightarrow 0, \tag{1.5}$$

where $\alpha(m) = \sum_\rho \langle m, n_\rho\rangle D_\rho$ and β is the quotient map. Then the degree of a monomial x^D is defined to be $\deg(x^D) = \beta(D)$. This graded ring is the *homogeneous coordinate ring* of X_Σ.

One can prove that $A_{n-1}(X_\Sigma)$ is the *Chow group* of Weil divisors modulo algebraic equivalence. To see how this relates to (1.5), note that (1.6) implies that

$$\operatorname{div}(\chi^m) = \sum_\rho \langle m, n_\rho\rangle D_\rho. \tag{1.6}$$

is linearly equivalent to zero. This explains the map α in (1.5). We should also mention that when X_Σ is smooth, $A_{n-1}(X_\Sigma)$ is the *Picard group* $\operatorname{Pic}(X_\Sigma)$.

Example 6.1 For $\mathbb{P}^n$, this construction gives $S = \mathbb{C}[x_0,\ldots,x_n]$ with the usual grading,

Example 6.2 For $\mathbb{P}^1\times\mathbb{P}^1$, Example 4.7 shows that we have divisors D_1, D_2 corresponding to the horizontal rays and divisors D_3, D_4 corresponding to vertical ones. If the corresponding variables are x_1, x_2, x_3, x_4, then we get the ring $S = \mathbb{C}[x_1,x_2,x_3,x_4]$. One can show that the Chow group is $\mathbb{Z}^2$ and that

$$\deg(x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}) = (a_1+a_2, a_3+a_4),$$

which is precisely the usual bigrading on $\mathbb{C}[x_1,x_2;x_3,x_4]$, where each graded piece consists of bihomogeneous polynomials in x_1, x_2 and x_3, x_4.

Example 6.2 generalies to $\mathbb{P}^n\times\mathbb{P}^m$, where $S = \mathbb{C}[x_0,\ldots,x_n;y_0,\ldots,y_m]$ with the usual bigrading.

We next use the variables x_ρ to give coordinates on X_Σ. To do this, we need an analog of the "irrelevant" ideal $\langle x_0,\ldots,x_n\rangle \subset \mathbb{C}[x_0,\ldots,x_n]$. For each cone $\sigma\in\Sigma$, let $x^{\hat\sigma}$ be the monomial

$$x^{\hat\sigma} = \prod\nolimits_{\rho\in\Sigma(1)\setminus\sigma(1)} x_\rho,$$

and then define the ideal $B \subset S$ to be

$$B = \big\langle x^{\hat\sigma} \mid \sigma\in\Sigma\big\rangle.$$

For $\mathbb{P}^n$, the reader should check that $B = \langle x_0,\ldots,x_n\rangle$.

The basic idea is that X_Σ should be a quotient of $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)$, where $\mathbf{V}(B) \subset \mathbb{C}^{\Sigma(1)}$ is the variety of B. The quotient is by the group G, where

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*).$$

Note that applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to (1.5) gives the exact sequence

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T. \tag{1.7}$$

This shows that G acts naturally on $\mathbb{C}^{\Sigma(1)}$ and leaves $\mathbf{V}(B)$ invariant.

The following representation of X_Σ was discovered by several people in the early 1990s.

Theorem 6.3 *Assume that X_Σ is a toric variety such that $\Sigma(1)$ spans $N_{\mathbb{R}}$. Then:*

(a) *X_Σ is the universal categorical quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$.*

(b) *X_Σ is a geometric quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$ if and only if X_Σ is simplicial.*

In the situation of the theorem, (1.7) is a short exact sequence, so that $T = (\mathbb{C}^*)^{\Sigma(1)}/G$. Thus

$$T = (\mathbb{C}^*)^{\Sigma(1)}/G \subset \big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G.$$

Furthermore, since the "big" torus $(\mathbb{C}^*)^{\Sigma(1)}$ acts naturally on $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)$, it follows that T acts on X_Σ. Quotients preserve normality, so that all of the requirements of being a toric variety are satisfied by the quotient in Theorem 6.3. In fact, one can *define* X_Σ to be the quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$.

Here are some examples of Theorem 6.3.

Example 6.4 For $\mathbb{P}^n$, the theorem gives the usual quotient representation $\mathbb{P}^n \simeq (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$.

Example 6.5 Continuing our example of $\mathbb{P}^1 \times \mathbb{P}^1$, we have $B = \langle x_1x_3, x_1x_4, x_2x_3, x_2x_4 \rangle$. Then, thinking of $\mathbb{C}^{\Sigma(1)}$ as $\mathbb{C}^2 \times \mathbb{C}^2$, one has

$$\mathbf{V}(B) = (\{0\} \times \mathbb{C}^2) \cup (\mathbb{C}^2 \times \{0\}).$$

One can also check that $G \simeq (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2 \times \mathbb{C}^2$ via

$$(\lambda, \mu) \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \mu x_3, \mu x_4).$$

Hence the quotient of Theorem 6.3 becomes

$$\Big(\mathbb{C}^2 \times \mathbb{C}^2 \setminus \big((\{0\} \times \mathbb{C}^2) \cup (\mathbb{C}^2 \times \{0\})\big)\Big)/(\mathbb{C}^*)^2,$$

which is exactly the way one usually represents $\mathbb{P}^1 \times \mathbb{P}^1$ as a quotient.

Example 6.6 We will construct global coordinates for the blow-up of $0 \in \mathbb{C}^n$. Let $e_1, \ldots, e_n$ be a basis of $N = \mathbb{Z}^n$, and let $\sigma = \mathbb{R}^n_+$ be the cone they generate. The resulting affine toric variety is $\mathbb{C}^n$. Then set $e_0 = e_1 + \cdots + e_n$ and consider the fan Σ whose cones are generated by all proper subsets of $\{e_0, \ldots, e_n\}$, excluding $\{e_1, \ldots, e_n\}$. We will prove that X_Σ is the blow-up of $0 \in \mathbb{C}^n$ using the representation of X_Σ given by Theorem 6.3.

If x_i corresponds to the edge generated by e_i, then the reader should show that the homogeneous coordinate ring of X_Σ is $\mathbb{C}[x_0,\ldots,x_n]$ where $\deg(x_0) = -1$ and $\deg(x_i) = +1$ for $1 \le i \le n$. Furthermore, $\mathbf{V}(B) = \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^n$ and $G = \mathbb{C}^*$ acts on $\mathbb{C}^{\Sigma(1)} = \mathbb{C} \times \mathbb{C}^n$ by $\mu \cdot (x_0, \mathbf{x}) = (\mu^{-1} x_0, \mu \mathbf{x})$. Then, given $(x_0, \mathbf{x}) \in \mathbb{C} \times \mathbb{C}^n \setminus \mathbf{V}(B)$, we can act on this point using G to obtain

$$\begin{array}{ll} (x_0,\mathbf{x}) \sim_G (1, x_0\mathbf{x}) & \text{if } x_0 \neq 0 \\ (0,\mathbf{x}) \sim_G (0, \mu\mathbf{x}) & \text{if } \mu \neq 0. \end{array}$$

In the first line, note that $\mathbf{x} \neq 0$, so that this part of the quotient is $\mathbb{C}^n \setminus \{0\}$. In the second line, we clearly get $\mathbb{P}^{n-1}$. Note also that the map $X_\Sigma \to \mathbb{C}^n$ given by $(x_0, x_1, \ldots, x_n) \mapsto (x_0x_1, \ldots, x_0x_n)$ is well-defined since $x_0 x_i$ has degree 0 and hence is invariant under the group action. It follows that X_Σ is the blow-up of $0 \in \mathbb{C}^n$.

If σ is an n-dimensional cone in $N_\mathbb{R}$, then the representation of X_σ given by Theorem 6.3 is especially simple in two cases:

- For σ smooth, the theorem gives $\mathbb{C}^n \simeq X_\sigma$.
- For σ simplicial, the theorem gives $\mathbb{C}^n/G \simeq X_\sigma$, where G is the finite group $N/(\oplus_\rho \mathbb{Z} n_\rho)$.

However, in the nonsimplicial case, things can be more complicated.

Example 6.7 For the cone σ of Example 4.3, Theorem 6.3 gives $\mathbb{C}^4/\!/G \simeq X_\sigma = \mathbf{V}(xy - zw)$, where $G = \mathbb{C}^*$ acts on $\mathbb{C}^4$ via

$$\lambda \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^{-1} x_3, \lambda^{-1} x_4).$$

The quotient is written $\mathbb{C}^4/\!/G$ because it is not a quotient in the usual group theoretic sense. To see why, consider the map $\mathbb{C}^4 \to X_\sigma$ given by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1x_3, x_2x_4, x_1x_4, x_2x_3).$$

If $p \in X_\sigma$, then one can show that

- $p \neq 0 \Rightarrow \pi^{-1}(p)$ is a G-orbit.
- $p = 0 \Rightarrow \pi^{-1}(p) = (\mathbb{C}^2 \times \{0\}) \cup (\{0\} \times \mathbb{C}^2)$.

7 Third Construction: Toric Ideals

Our third construction involves toric ideals. Let's begin with a special case. Let $\sigma \subset N_\mathbb{R}$ be a cone, and suppose that $\mathscr{A} = \{m_1, \ldots, m_s\}$ generates the semigroup $\sigma^\vee \cap M$. The map sending $y_i \mapsto \chi^{m_i}$ gives a surjective homomorphism $\mathbb{C}[y_1, \ldots, y_s] \to \mathbb{C}[\sigma^\vee \cap M]$. The kernel $I_\mathscr{A}$ is a *toric ideal*.

A key observation is that $I_\mathscr{A}$ is generated by *binomials* (a binomial is a difference to two monomials). To state this precisely, note that $\alpha = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ can be uniquely written $\alpha = \alpha^+ - \alpha^-$, where α^+ and α^- have nonnegative entries and disjoint support. Then the toric ideal $I_\mathscr{A} \subset \mathbb{C}[y_1, \ldots, y_s]$ is

$$I_\mathscr{A} = \langle y^{\alpha^+} - y^{\alpha^-} \mid \alpha = (a_1, \ldots, a_s) \in \mathbb{Z}^s,\ \textstyle\sum_{i=1}^s a_i m_i = 0 \rangle. \tag{1.8}$$

In practice, toric ideals are defined in much greater generality and are closely related to nonnormal toric varieties. To set this up, let $\mathscr{A} = \{m_1, \dots, m_s\}$ be *any* finite subset of $\mathbb{Z}^n$. Then define the *toric ideal* $I_{\mathscr{A}}$ using the right-hand side of (1.8). Toric ideals are easy to characterize: an ideal in $\mathbb{C}[y_1, \dots, y_s]$ is a toric ideal $I_{\mathscr{A}}$ if and only if it is prime and is generated by binomials.

Thinking geometrically, $I_{\mathscr{A}}$ defines a subvariety $X_{\mathscr{A}} \subset \mathbb{C}^s$. One can show that $X_{\mathscr{A}}$ is the Zariski closure of the image of the map $(\mathbb{C}^*)^n \to \mathbb{C}^s$ defined by

$$t \mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)). \tag{1.9}$$

Note also that $X_{\mathscr{A}}$ contains a torus (the image of $(\mathbb{C}^*)^n$ under the map (1.9)). Hence $X_{\mathscr{A}}$ satisfies all of the criteria for being a toric variety, except possibly normality. For this reason, we call $X_{\mathscr{A}}$ a *generalized affine toric variety*. Here are two facts about $X_{\mathscr{A}}$:

- $X_{\mathscr{A}}$ is a toric variety in the usual sense (i.e., is normal) if and only if $\mathbb{N}\mathscr{A} = \mathrm{Cone}(\mathscr{A}) \cap \mathbb{Z}\mathscr{A}$, where $\mathrm{Cone}(\mathscr{A})$ is the cone generated by $\mathscr{A}$, and $\mathbb{Z}\mathscr{A}$ (resp. $\mathbb{N}\mathscr{A}$) is the set of all integer (resp. nonnegative integer) combinations of elements of $\mathscr{A}$.
- The normalization of $X_{\mathscr{A}}$ is the affine toric variety X_σ, where $\sigma \subset N_{\mathbb{R}}$ is the cone dual to $\mathrm{Cone}(\mathscr{A})$ and N is the dual of $\mathbb{Z}\mathscr{A}$.

Example 7.1 The toric variety of Example 4.3 is $X_{\mathscr{A}}$, where $\mathscr{A} = \{m_1, m_2, m_3, m_4\}$ as in (1.4).

Example 7.2 Given $\mathscr{A} = \{\beta_1, \dots, \beta_s\} \subset \mathbb{Z}$, we get a *monomial curve* in $\mathbb{C}^s$ parametrized by

$$t \mapsto (t^{\beta_1}, \dots, t^{\beta_s}).$$

Since t^{β_i} is a character on $\mathbb{C}^*$, this is the generalized affine toric variety $X_{\mathscr{A}}$. It is nonnormal precisely when $X_{\mathscr{A}}$ fails to be smooth. The simplest example is the cusp parametrized by $t \mapsto (t^2, t^3)$. Here, the corresponding toric ideal is generated by the binomial $y^2 - x^3$.

Besides the generalized affine toric variety $X_{\mathscr{A}} \subset \mathbb{C}^s$, we also get a projective variety $Y_{\mathscr{A}} \subset \mathbb{P}^{s-1}$ by regarding (1.9) as a map $T \to \mathbb{P}^{s-1}$. More precisely, the *generalized projective toric variety* $Y_{\mathscr{A}}$ is defined to be the Zariski closure of the image of this map. Generalized projective toric varieties arise naturally in many different contexts.

Example 7.3 Suppose that $\mathscr{A} = \{m_1, \dots, m_s\} \subset \mathbb{Z}^n$ and that $\mathscr{A}$ generates $\mathbb{Z}^n$. Then let $L(\mathscr{A})$ be the set of Laurent polynomials with exponent vectors in $\mathscr{A}$, i.e.,

$$L(\mathscr{A}) = \left\{ a_1 \mathbf{t}^{m_1} + \cdots + a_s \mathbf{t}^{m_s} \mid a_i \in \mathbb{C} \right\},$$

where $\mathbf{t}^m = t_1^{a_1} \cdots t_n^{a_n}$ for $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Given $n+1$ Laurent polynomials $f_0, \dots, f_n \in L(\mathscr{A})$, their *$\mathscr{A}$-resultant*

$$\mathrm{Res}_{\mathscr{A}}(f_0, \dots, f_n)$$

is a polynomial in the coefficients of the f_i whose vanishing is necessary and sufficient for the equations $f_0 = \cdots = f_n = 0$ to have a solution. However, one must be careful where the solution lies. The f_i are defined initially on the torus $(\mathbb{C}^*)^n$, but the definition of generalized projective toric variety shows that the equation $f_i = 0$ makes sense on $Y_{\mathscr{A}}$. Then one can prove that

$$\mathrm{Res}_{\mathscr{A}}(f_0, \dots, f_n) = 0 \iff f_1 = \cdots = f_n = 0 \text{ have a solution in } Y_{\mathscr{A}}.$$

Generalized toric varieties and toric ideals also have applications to hypergeometric equations and combinatorics.

Lecture II: Toric Varieties and Polytopes

1 The Toric Variety of a Polytope

Let $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$ be as in Lecture I. A *lattice polytope* Δ in $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ is the convex hull of a finite subset of M. We will show that an n-dimensional lattice polytope Δ determines a projective toric variety X_Δ.

To do this, we first describe Δ. For each facet F of Δ, there is an inward normal primitive vector $n_F \in N$ and integer a_F such that

$$\Delta = \{m \in M_\mathbb{R} \mid \langle m, n_F \rangle \geq -a_F \text{ for all facets } F\}. \tag{2.1}$$

Given any face $\mathscr{F}$ of Δ, let $\sigma_\mathscr{F}$ be the cone generated by n_F for all facets F containing $\mathscr{F}$. Then

$$\Sigma_\Delta = \{\sigma_\mathscr{F} \mid \mathscr{F} \text{ is a face of } \Delta\}$$

is a complete fan which is called the *normal fan* of Δ. This gives a toric variety denoted X_Δ.

Example 1.1 The unit square $\square$ with vertices $(0,0),(1,0),(1,1),(0,1)$ can be represented

$$\square = \{a \geq 0\} \cap \{-a \geq -1\} \cap \{b \geq 0\} \cap \{-b \geq -1\}.$$

It follows that the inward normals are $\pm e_1$ and $\pm e_2$. This gives the following normal fan:

- $\pm e_1$ and $\pm e_2$ generate the 1-dimensional cones of the normal fan.
- The vertex $(0,1)$ lies in the faces defined by $a = 0$ and $-b = -1$ corresponding to inner normals e_1 and $-e_2$. Hence this face corresponds to the cone generated by e_1 and $-e_2$.

The other vertices are handled similarly, and the resulting normal fan is the one appearing in Example 4.7. Hence $X_\square = \mathbb{P}^1 \times \mathbb{P}^1$.

In general, we can characterize these fans as follows.

Theorem 1.2 *The toric variety of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$ is projective if and only if Σ is the normal fan of an n-dimensional lattice polytope in $M_\mathbb{R}$.*

We should also note that the polytope Δ is combinatorially dual to it normal fan Σ_Δ. This means that there is a one-to-one inclusion reversing correspondence

$$\sigma_\mathscr{F} \in \Sigma_\Delta \longleftrightarrow \mathscr{F} \subset \Delta$$

between cones of Σ_Δ and faces of Δ (provided we count Δ as a face of itself) such that

$$\dim \sigma_\mathscr{F} + \dim \mathscr{F} = n$$

for all faces $\mathscr{F}$ of Δ. Combining this with the correspondence between cones of Σ_Δ and torus orbits from Lecture I gives a one-to-one dimension preserving correspondence between faces of Δ and torus orbits of X_Δ. Thus Δ determines the combinatorics of the toric variety X_Δ.

In particular, a facet F of Δ corresponds to the edge σ_F Σ_Δ generated by n_F. This in turn corresonds to a divisor D_F on X_Δ. Then the representation (2.1) gives the divisor

$$D_\Delta = \sum_F a_F D_F.$$

Then one can show that

$$H^0(X_\Delta, \mathscr{O}_{X_\Delta}(D_\Delta)) = \bigoplus_{m\in\Delta\cap M} \mathbb{C}\cdot\chi^m. \tag{2.2}$$

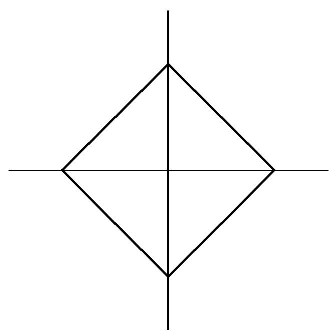
If we write $\Delta\cap M = \{m_1,\ldots,m_s\}$, then the sections χ^{m_i} give the map (1.9) defined by

$$t \mapsto (\chi^{m_1}(t),\ldots,\chi^{m_s}(t)),$$

which extends to a map $X_\Delta \hookrightarrow \mathbb{P}^{s-1}$. In fact, for $\nu \gg 0$, the corresponding map for $\nu\Delta$ is an embedding (this is how one proves Theorem 1.2), so that in notation of Lecture I, $X_{\nu\Delta}$ is the projective variety $Y_{\mathscr{A}}$, where $\mathscr{A} = \nu\Delta\cap M$ be the set of lattice points in $\nu\Delta$. Also, $X_{\nu\Delta} = X_\Delta$ since Δ and $\nu\Delta$ have the same normal fan.

There is also a dual version of this construction. Suppose that $P \subset N_\mathbb{R}$ is an n-dimensional polytope which contains the origin as an interior point and whose vertices lie in $\mathbb{Q}^n$. Then we get a complete fan Σ_P in $N_\mathbb{R}$ by taking cones (relative to the origin) over the faces of P. The resulting toric variety is denoted X_P.

Example 1.3 Consider the tilted square P in the plane:



The fan Σ_P obtained by taking cones over faces is the fan of Example 4.7. Hence $X_P = \mathbb{P}^1\times\mathbb{P}^1$.

To see how this relates to our earlier construction, we define the *polar* or *dual* of $P\subset N_\mathbb{R}$ to be

$$P^\circ = \{m\in M_\mathbb{R} \mid \langle m,u\rangle \geq -1 \text{ for all } u\in P\}.$$

Since P has rational vertices, so does P°, which means that $\Delta = \ell P^\circ$ is a lattice polytope for some positive integer ℓ. Then one can show that Σ_P is the normal fan of Δ, so that X_P is the projective toric variety X_Δ.

A quite different method for constructing X_Δ is due to Batyrev. Given Δ, consider the cone over $\Delta\times\{1\}\subset M_\mathbb{R}\oplus\mathbb{R}$. The integer points of the cone give a semigroup algebra S_Δ. Since $(m,k)\in M\oplus\mathbb{Z}$ is in the cone if and only if $m\in k\Delta$, S_Δ is the subring of $\mathbb{C}[t_0,t_1^{\pm1},\ldots,t_n^{\pm1}]$ spanned by Laurent monomials $t_0^k\mathbf{t}^m$ with $k\geq 0$ and $m\in k\Delta$. This ring can be graded by setting $\deg(t_0^k\mathbf{t}^m) = k$, and one can show that

$$X_\Delta = \mathrm{Proj}(S_\Delta).$$

Since S_Δ is the coordinate ring of an affine toric variety, it is Cohen-Macaulay and hence X_Δ is arithmetically Cohen-Macaulay.

2 The Dehn-Sommerville Equations

Euler's formula for a 3-dimensional polytope Δ in $\mathbb{R}^3$ is

$$f_0 - f_1 + f_2 = 2, \tag{2.3}$$

where f_i is the number of i-dimensional faces of Δ. If Δ has the additional property that all of its facets are triangles (such as a tetrahedron, octahedron or icosahedron), then counting edges gives

$$3f_2 = 2f_1. \tag{2.4}$$

To generalize these, suppose that P is an n-dimensional polytope in $\mathbb{R}^n$ such that every facet is *simplicial*, meaning that every facet has exactly n vertices. For such a polytope, let f_i be the number of i-dimensional faces of P, and let $f_{-1} = 1$. Then, for $0 \leq p \leq n$, set

$$h^p = \sum_{i=p}^{n} (-1)^{i-p} \binom{i}{p} f_{n-i-1}.$$

The *Dehn-Sommerville* equations assert that if $P \subset \mathbb{R}^n$ is an n-dimensional simplicial polytope, then

$$h^p = h^{n-p} \quad \text{for all } 0 \leq p \leq n. \tag{2.5}$$

When $n = 3$, (2.3) is $h^0 = h^3$ and (2.4) is equivalent to $h^1 = h^2$ (assuming $h^0 = h^3$).

To prove (2.5), note that we can move P so that the origin is an interior point. Furthermore, wiggling the vertices by a small amount does note change the combinatorial type of P. Thus we may assume that its vertices lie in $\mathbb{Q}^n$. Then, as in the previous section, projecting from origin to the faces of P gives a fan in $N_{\mathbb{R}} = \mathbb{R}^n$ which is simplicial since P is. This gives a projective toric variety X_P.

Being projective and simplicial implies two nice facts about X_P:

- $h^p = \dim H^{2p}(X_P, \mathbb{Q})$ for $0 \leq p \leq n$.
- Poincaré Duality holds for X_P, i.e., $\dim H^q(X_P, \mathbb{Q}) = \dim H^{2n-q}(X_P, \mathbb{Q})$ for $0 \leq q \leq 2n$.

The Dehn-Sommerville equations (2.5) follow immediately!

In the smooth case, the second bullet is Poincaré Duality. For the first bullet, note that $X_{\mathbb{P}}$ is a union of affine toric varieties $X_\sigma \simeq \mathbb{C}^n$. Then the formula for $\dim H^{2p}(X_P, \mathbb{Q})$ follows straightforwardly since

$$X_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$$

when σ is a smooth cone of dimension k. The simplicial case is similar since an orbifold is a rational homology manifold.

This is very pretty but is not the end of the story. One can also ask if it is possible to characterize *all* possible vectors $(f_0, f_1, \ldots, f_{n-1})$ coming from n-dimensional simplicial polytopes. For example, when $n = 3$, one can show that a vector of positive integers (f_0, f_1, f_2) comes from a 3-dimensional simplicial polytope if and only if $f_0 \geq 4$ and (2.3) and (2.4) are satisfied. This can be generalized to arbitrary dimensions, though the result takes some work to state. A nice account can be found in Section 5.6 of Fulton's book. What's interesting is that the proof uses the Hard Lefschetz Theorem for simplicial toric varieties (which is a very difficult theorem).

3 The Ehrhart Polynomial

This section will use toric varieties to prove the following wonderful result of Ehrhart concerning lattice points in integer multiples of lattice polytopes.

Theorem 3.1 *Let Δ be an n-dimensional lattice polytope in $M_{\mathbb{R}} = \mathbb{R}^n$. Then there is a unique polynomial E_Δ* (*the* **Ehrhart polynomial**) *with coefficients in $\mathbb{Q}$ which has the following properties:*

(a) *For all integers $\nu \geq 0$,*

$$E_\Delta(\nu) = \#(\nu\Delta \cap M)$$

(b) *If the volume is normalized so that the unit n-cube determined by a basis of M has volume 1, then the leading coefficient of E_Δ is* $\mathrm{vol}(\Delta)$.

(c) *If* $\mathrm{int}(\Delta)$ *is the interior of Δ, then the* **reciprocity law** *states that for all integers $\nu > 0$,*

$$E_\Delta(-\nu) = (-1)^n \#(\nu\,\mathrm{int}(\Delta) \cap M).$$

Before giving the proof, let's give a classic application in dimension 2. If Δ is a lattice polygon, then the Ehrhart polynomial is

$$E_\Delta(x) = \mathrm{area}(\Delta)\, x^2 + Bx + 1 \tag{2.6}$$

since $E_\Delta(0) = \#(0 \cdot \Delta \cap M) = 1$. If we let $\partial\Delta$ denote the boundary of Δ, then

$$E_\Delta(1) = \#(\Delta \cap M) = \#(\mathrm{int}(\Delta) \cap M) + \#(\partial\Delta \cap M) = E_\Delta(-1) + \#(\partial\Delta \cap M),$$

where the last equality uses the reciprocity law. By (2.6), we also have

$$E_\Delta(1) = \mathrm{area}(\Delta) + B + 1 \quad \text{and} \quad E_\Delta(-1) = \mathrm{area}(\Delta) - B + 1.$$

Combining these equalities gives the following:

- $B = \frac{1}{2}\#(\partial\Delta \cap M)$, so that the Ehrhart polynomial of a lattice polygon is

$$E_\Delta(x) = \mathrm{area}(\Delta)\, x^2 + \tfrac{1}{2}\#(\partial\Delta \cap M)\, x + 1.$$

- In particular, setting $x = 1$ gives *Pick's Formula*

$$\#(\Delta \cap M) = \mathrm{area}(\Delta) + \tfrac{1}{2}\#(\partial\Delta \cap M) + 1.$$

We now turn to the proof of Theorem 3.1. While this result can be proved by elementary means, we will give a proof which uses the cohomology of line bundles on the toric variety X_Δ. Recall that in Section 1 we represented Δ as the intersection (2.1). We also had the line bundle

$$L = \mathcal{O}_{X_\Delta}(D_\Delta), \quad D_\Delta = \sum_F a_F D_F.$$

By (2.2), the global sections of L are

$$H^0(X_\Delta, L) = \bigoplus_{m \in \Delta \cap M} \mathbb{C} \cdot \chi^m. \tag{2.7}$$

To prove Theorem 3.1, we first consider the *Euler-Poincaré characteristic*

$$\chi(X_\Delta, L) = \sum_{i=0}^{n} (-1)^i \dim H^i(X_\Delta, L).$$

By a result of Kleiman, there is a polynomial h_L of degree at most n such that

$$\chi(X_\Delta, L^{\otimes \nu}) = h_L(\nu) \tag{2.8}$$

for *all* integers ν.

However, the line bundle L is ample—this is part of the proof that X_Δ is projective. This has nice consequences for the Euler-Poincaré characteristic. First, on any complete variety, we have:

- Any positive tensor power of an ample line bundle is ample.

Furthermore, line bundles on complete toric varieties have the following special properties:

- An ample line bundle on a complete toric variety is generated by its global sections.
- Let $\mathscr{L}$ be a line bundle on a complete toric variety X. If $\mathscr{L}$ is generated by its global sections, then $H^i(X, \mathscr{L}) = 0$ for all $i > 0$.

These three bullets and the ampleness of L imply that

$$H^i(X_\Delta, L^{\otimes \nu}) = 0$$

when $i > 0$ and $\nu \geq 0$.[1] Using this, the Euler-Poincaré characteristic simplifies to

$$\chi(X_\Delta, L^{\otimes \nu}) = \dim H^0(X_\Delta, L^{\otimes \nu})$$

when $\nu \geq 0$, and combining this with (2.8), we conclude that the polynomial h_L satisfies

$$\dim H^0(X_\Delta, L^{\otimes \nu}) = h_L(\nu) \tag{2.9}$$

for all $\nu \geq 0$.

The next observation is that the polytopes Δ and $\nu\Delta$ give the same normal fan and hence the same toric varieties, i.e., $X_\Delta = X_{\nu\Delta}$. Furthermore, the divisor associated to $\nu\Delta$ is

$$D_{\nu\Delta} = \nu D_\Delta,$$

which means that the associated ample line bundle is $L^{\otimes \nu}$. It follows that if we apply (2.7) with $L^{\otimes \nu}$ in place of L, then we obtain

$$H^0(X_\Delta, L^{\otimes \nu}) = \bigoplus_{m \in \nu\Delta \cap M} \mathbb{C} \cdot \chi^m.$$

Combining this with (2.9), we conclude that

$$\#(\nu\Delta \cap M) = \dim H^0(X_\Delta, L^{\otimes \nu}) = h_L(\nu)$$

for all $\nu \geq 0$. This shows that $E_\Delta = h_L$ satisfies the first assertion of the theorem.

[1] When $\nu = 0$, note that $L^{\otimes \nu} = \mathscr{O}_{X_\Delta}$ is generated by global sections.

The second follows easily from the first, for if $E_\Delta(x) = a_n x^n + \cdots + a_0$, then

$$a_n = \lim_{\nu\to\infty} \frac{E_\Delta(\nu)}{\nu^n} = \lim_{\nu\to\infty} \frac{\#(\nu\Delta \cap M)}{\nu^n} = \mathrm{vol}(\Delta).$$

The proof of the third assertion is more sophisticated. Recall from Theorem 5.1 of Lecture I that the dualizing sheaf of X_Δ is

$$\omega_{X_\Delta} = \mathscr{O}_{X_\Delta}(-\sum_F D_F),$$

where as usual D_F is the divisor corresponding to the facet F of Δ. Since X_Δ is Cohen-Macaulay, *Serre Duality* implies that

$$H^i(X_\Delta, L^{\otimes(-\nu)}) \simeq H^{n-i}(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta})^*.$$

In terms of the Euler-Poincaré characteristic, this easily implies

$$\chi(X_\Delta, L^{\otimes(-\nu)}) = (-1)^n \chi(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta}).$$

If we combine this with (2.8), we see that the Ehrhart polynomial $E_\Delta = h_L$ satisfies

$$E_\Delta(-\nu) = (-1)^n \chi(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta})$$

for all ν. But if $\nu > 0$, then $L^{\otimes \nu}$ is ample, so that the *Kodaira Vanishing Theorem* implies that

$$H^i(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta}) = 0$$

when $\nu > 0$. Hence, for these ν, the above formula simplifies to

$$E_\Delta(-\nu) = (-1)^n \dim H^0(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta}).$$

The final step in the proof is to show that

$$\dim H^0(X_\Delta, L^{\otimes \nu} \otimes \omega_{X_\Delta}) = \#(\nu\,\mathrm{int}(\Delta) \cap M)$$

for $\nu > 0$. By our trick of replacing Δ with $\nu\Delta$, it suffices to prove this for $\nu = 1$. Note that

$$L \otimes \omega_{X_\Delta} = \mathscr{O}_{X_\Delta}\big(D_\Delta - \textstyle\sum_F D_F\big) = \mathscr{O}_{X_\Delta}\big(\textstyle\sum_F (a_F - 1) D_F\big),$$

where $\Delta = \bigcap_F \{m \in M_{\mathbb{R}} \mid \langle m, n_F\rangle \geq -a_F\}$. Since

$$\mathrm{int}(\Delta) \cap M = \bigcap_F \{m \in M \mid \langle m, n_F\rangle > -a_F\} = \bigcap_F \{m \in M \mid \langle m, n_F\rangle \geq -(a_F - 1)\},$$

the methods used to prove (2.7) imply that

$$H^0\big(X_\Delta, \mathscr{O}_{X_\Delta}(\textstyle\sum_F (a_F - 1) D_F)\big) = \displaystyle\bigoplus_{m \in \mathrm{int}(\Delta)\cap M} \mathbb{C} \cdot \chi^m.$$

This completes the proof of the theorem!

We conclude this section with another application of the Ehrhart polynomial. Given a finite set $\mathscr{A} = \{m_1, \dots, m_s\} \subset \mathbb{Z}^n$, we get the (possibly nonnormal) projective toric variety $Y_{\mathscr{A}} \subset \mathbb{P}^{s-1}$ defined in Section 7 of Lecture I. We now give a criterion for $Y_{\mathscr{A}}$ to be normal which involves the Ehrhart polynomial of the polytope $\Delta = \mathrm{Conv}(\mathscr{A})$.

To state the criterion, we define the *Hilbert polynomial* of $Y_{\mathscr{A}}$ to be the unique polynomial $H_{\mathscr{A}}$ for which

$$H_{\mathscr{A}}(\nu) = \#\{m_{i_1} + \cdots + m_{i_\nu} \mid m_{i_1}, \dots, m_{i_\nu} \in \mathscr{A}\}$$

for $\nu \gg 0$. One can show that the polynomials $H_{\mathscr{A}}$ and E_Δ have the same leading term, which is the normalized volume of Δ. Then we have the following result of Sturmfels.

Theorem 3.2 *The toric variety $Y_{\mathscr{A}} \subset \mathbb{P}^{\ell-1}$ is normal if and only if the Hilbert polynomial $H_{\mathscr{A}}$ equals the Ehrhart polynomial E_Δ.*

4 The BKK Theorem

So far, we have used the number of faces of a polytope (in the Dehn-Sommerville equations) and the number of lattice points (in the Ehrhart polynomial). But what about the volume of the polytope? This plays in subsidiary role in the Ehrhart polynomial. It is now time for the volume to take a more central role.

We will begin with a rather special situation. Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. Then consider n Laurent polynomials

$$f_i = \sum_{m \in \Delta \cap M} c_{i,m} \mathbf{t}^m \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \quad 1 \le i \le n,$$

where $\mathbf{t}^m = t_1^{a_1} \cdots t_n^{a_n}$, $m = (a_1, \dots, a_n)$, is the character denoted χ^m in Lecture I.

Theorem 4.1 *If $f_1, \dots, f_n$ as above are generic, then the equations*

$$f_1 = \cdots = f_n = 0$$

have $n!\mathrm{vol}(\Delta)$ solutions in $(\mathbb{C}^*)^n$.

To prove this, we will work on the toric variety X_Δ. By (2.7), the polynomials f_i are global sections of the ample line bundle $L = \mathscr{O}_{X_\Delta}(D_\Delta)$. This means that for generic sections, we have:

- The number of solutions of $f_1 = \cdots = f_n = 0$ is finite.
- We can assume that the solutions lie in the torus $(\mathbb{C}^*)^n \subset X_\Delta$.
- The number of solutions is the n-fold intersection number $(D_\Delta)^n$.

Concerning D_Δ^n, we note that since X_Δ is a possibly singular variety, we can't use the usual definition of intersection number coming from cup product on cohomology. Instead, we use Kleiman's intersection theory for normal varieties, which in this case implies that

$$\dim H^0(X_\Delta, L^{\otimes \nu}) = \frac{(D_\Delta)^n}{n!} \nu^n + \text{lower order terms in } \nu$$

since L is ample. However, in the previous section, we showed that

$$\dim H^0(X_\Delta, L^{\otimes \nu}) = E_\Delta(\nu) = \mathrm{vol}(\Delta)\, \nu^n + \text{lower order terms in } \nu.$$

This shows that

$$(D_\Delta)^n = n!\mathrm{vol}(\Delta).$$

Since this gives the number of solutions of $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}^*)^n$, the theorem is proved!

Theorem 4.1 can be generalized considerably. Suppose instead that we have arbitrary Laurent polynomials

$$f_1, \ldots, f_n \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

Then let Δ_i be the *Newton polytope* of f_i, which means that Δ_i is the convex hull of the exponent vectors of the nonzero terms of f_i. In this situation, one can define the *mixed volume*

$$MV_n(\Delta_1, \ldots, \Delta_n).$$

This is discussed in Section 5.4 of Fulton's book. Then the *Bernstein-Kushnirenko-Khovanskii Theorem* is as follows:

Theorem 4.2 *Consider the solutions in $(\mathbb{C}^*)^n$ of the equations*

$$f_1 = \cdots = f_n = 0.$$

(a) *If there are only finitely many solutions, then the number of solutions is bounded above by $MV_n(\Delta_1, \ldots, \Delta_n)$.*

(b) *If the f_i have generic coefficients, then the number of solutions equals $MV_n(\Delta_1, \ldots, \Delta_n)$.*

One of the properties of the mixed volume is that when all of the polytopes are the same, then

$$MV_n(\Delta, \ldots, \Delta) = n!\mathrm{vol}(\Delta).$$

It follows that Theorem 4.1 is a special case of the BKK theorem.

5 Reflexive Polytopes and Fano Toric Varieties

The dualizing sheaf on $\mathbb{P}^n$ is easily seen to be

$$\omega_{\mathbb{P}^n} = \mathscr{O}_{\mathbb{P}^n}(-(n+1)),$$

which means that its dual $\mathscr{O}_{\mathbb{P}^n}(n+1)$ is ample. More generally, let V be a complete Cohen-Macaulay variety with dualizing sheaf ω_V. Then we say that V is *Fano* if the dual of ω_V is an ample line bundle.

The goal of this section is to characterize Fano toric varieties. First recall that the dualizing sheaf of a toric variety X is

$$\omega_X = \mathscr{O}_X(-\textstyle\sum_\rho D_\rho).$$

It is customary to call $K_X = -\sum_\rho D_\rho$ the *canonical divisor* of X. Thus being Fano means that the anticanonical divisor $-K_X = \sum_\rho D_\rho$ is ample.

Being Fano is a very special property. Hence, in order for the toric variety of a polytope to be Fano, the polytope needs to be rather special. This is where we encounter Batyrev's notion of a *reflexive polytope*. Here is the precise definition.

Definition 5.1 *A n-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is* **reflexive** *if the following two conditions hold:*

(a) *All facets F of Δ are supported by an affine hyperplane of the form $\{m \in M_{\mathbb{R}} \mid \langle m, n_F \rangle = -1\}$ for some $n_F \in N$.*

(b) $\operatorname{int}(\Delta) \cap M = \{0\}$.

Reflexive polytopes have a very pretty combinatorial duality. Let Δ be an lattice polytope, and let Δ° be the polar polytope defined in Section 1 of this lecture. Besides $(\Delta^\circ)^\circ = \Delta$, Batyrev showed that the basic duality between Δ and Δ° is as follows.

Lemma 5.2 *Δ is reflexive if and only if Δ° is reflexive.*

Reflexive polytopes are interesting in this context because of the following result, which characterizes Fano toric varieties.

Theorem 5.3 *A complete toric variety X is Fano if and only if there is a reflexive polytope Δ such that $X = X_\Delta$.*

To prove this, first assume that Δ is reflexive. Then the definition of reflexive implies that

$$\Delta = \bigcap_F \{m \in M_{\mathbb{R}} \mid \langle m, n_F \rangle \geq -1\}.$$

Thus $a_F = 1$ for all F, which means that the associated divisor is

$$D_\Delta = \sum_F a_F D_F = \sum_F D_F = -K_X.$$

We know that D_Δ is ample, which proves that X_Δ is Fano. The converse is equally easy, and the theorem is proved!

The simplest example of a Fano toric variety is $\mathbb{P}^n$. The next case to consider is weighted projective space, where the answer is slightly more interesting.

Lemma 5.4 *Let $X = \mathbb{P}(q_0, \ldots, q_n)$ be a weighted projective space, and let $q = \sum_{i=0}^n q_i$. Then X is Fano if and only if $q_i | q$ for all i.*

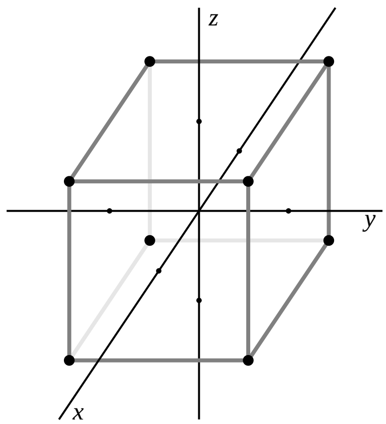
In any given dimension, there are only finitely many reflexive polytopes up to unimodular transformation, which means that there are only finitely many toric Fano varieties of dimension n up to isomorphism. Smooth toric Fano varieties have been classified in low dimensions, and attempts are underway to classify *all* 4-dimensional reflexive polytopes.

Since reflexive polytopes come in pairs Δ, Δ°, we get toric Fano varieties X_Δ, X_{Δ° which are in some sense dual. In Lecture III, we will use these toric varieties to create "dual" families of Calabi-Yau hypersurfaces which are important in mirror symmetry.

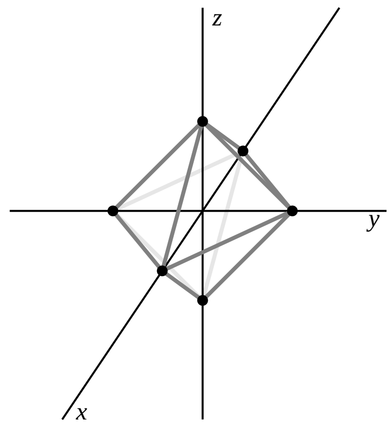
Here is an example of a reflexive polytope and its dual.

Example 5.5 Let $M = \mathbb{Z}^3$, and consider the cube $\Delta \subset M_\mathbb{R}$ centered at the origin with vertices $(\pm1, \pm1, \pm1)$. This gives the toric variety $X = X_\Delta$. To describe the fan of X, note that the polar $\Delta^\circ \subset N_\mathbb{R}$ is the octahedron with vertices $\pm e_1, \pm e_2, \pm e_3$. Thus the normal fan is formed from the faces of the octahedron, giving a fan Σ whose 3-dimensional cones are the octants of $\mathbb{R}^3$. It follows easily that $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We have the following pictures of Δ and Δ°:



$\Delta \subset M_{\mathbf{R}}$



$\Delta^\circ \subset N_{\mathbf{R}}$

It is easy to check that the cube $\Delta \subset M_\mathbb{R}$ and the octahedron $\Delta^\circ \subset N_\mathbb{R}$ are dual reflexive polytopes. In particular, Δ° gives a "dual" toric variety $X^\circ = X_{\Delta^\circ}$, which is determined by the normal fan of Δ° (= the fan in $M_\mathbb{R}$ formed by cones over the faces of the cube Δ). Hence we have a pair of "dual" toric varieties, X and X°. It is interesting to observe that X is smooth while X° is rather singular. In fact, the 3-dimensional cones of Σ° are not even simplicial—they're all infinite pyramids. However, since Δ and Δ° is reflexive, we know that X and X° are Fano.

Note that Δ and Δ° also differ with respect to lattice points. For $\Delta^\circ \subset N_\mathbb{R}$, the only lattice points in N are the origin and vertices, while $\Delta \subset M_\mathbb{R}$ has many more since the midpoints of the edges and the centers of the faces lie in M. This example also shows that we can start with a smooth toric variety (such as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) and by "duality" wind up with something singular. As we will see in Lecture III, this has implications for mirror symmetry.

Lecture III: Toric Varieties and Mirror Symmetry

1 The Quintic Threefold

In 1991, Candelas, de la Ossa, Green and Parkes made some startling predictions about rational curves on a generic quintic hypersurface $V \subset \mathbb{P}^4$ (the *quintic threefold*). In particular, they claimed that V contained the following numbers of rational curves:

- 2875 lines.
- 609250 conics.
- 317206375 cubics.
- 242467530000 quartics.

More generally, if we let

$$n_d = \text{“\# rational curves of degree } d \text{ in } V\text{”}$$

(we'll explain the quotation marks below), then Candelas et. al. gave the following receipe for n_d:

(a) The differential equation

$$\begin{aligned} 0 \;\; = \;\; & \Big(x\frac{d}{dx}\Big)^4 y + \frac{2\cdot 5^5 x}{1+5^5x}\Big(x\frac{d}{dx}\Big)^3 y + \frac{7\cdot 5^4 x}{1+5^5x}\Big(x\frac{d}{dx}\Big)^2 y \\ & + \frac{2\cdot 5^4 x}{1+5^5x}\Big(x\frac{d}{dx}\Big) y + \frac{24\cdot 5x}{1+5^5x} y. \end{aligned}$$

has a regular singular point at $x=0$ with maximally unipotent monodromy.

(b) Two solutions of this differential equation are

$$y_0(x) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}(-1)^n x^n$$

and

$$y_1 = y_0(x)\log(-x) + 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\Big[\sum_{j=n+1}^{5n}\frac{1}{j}\Big](-1)^n x^n.$$

(c) The differential equation

$$\Big(x\frac{d}{dx}\Big)Y = \frac{-5^5x}{1+5^5x}Y$$

has the solution

$$Y = \frac{c}{1+5^5x}, \quad c \text{ constant.}$$

(d) Finally, setting $c=5$ and $q = \exp(y_1(x)/y_0(x))$, we have

$$5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = \frac{5}{(1+5^5x)}\frac{1}{y_0(x)^2}\Big(\frac{q}{x}\frac{dx}{dq}\Big)^3.$$

2 Instanton Numbers

Making sense of the numbers n_d is a nontrivial task. To define these rigorously, the first steps are as follows:

- Define the moduli stack $\overline{M}_{0,0}(\mathbb{P}^4,d)$ of 0-pointed stable curves $f: C \to \mathbb{P}^4$ of genus 0 and degree d.
- Define the vector bundle $\mathscr{V}_d$ on $\overline{M}_{0,0}(\mathbb{P}^4,d)$ whose fiber at $f: C \to \mathbb{P}^4$ is $H^0(C, f^*\mathscr{O}_{\mathbb{P}^4}(5))$.

Using this, we define the *Gromov-Witten invariant* of the quintic threefold V to be

$$\langle I_{0,0,d} \rangle = \int_{\overline{M}_{0,0}(\mathbb{P}^4,d)} c_{\mathrm{top}}(\mathscr{V}_d).$$

In 1995, Kontsevich computed that

$$\langle I_{0,0,4} \rangle = \frac{15517926796875}{64}.$$

Gromov-Witten invariants can be defined for any smooth variety. One of the major unsolved problems in mirror symmetry is to understand the enumerative significance of these numbers.

For the quintic threefold, we can approach this problem using *instanton numbers* n_d, which are defined recursively by the equation

$$\langle I_{0,0,d} \rangle = \sum_{k|d} n_{d/k} k^{-3}.$$

Here are three important theorems about the instanton numbers of the quintic threefold.

Theorem 2.1 *(Givental,Lian/Liu/Yau) The instanton numbers n_d of the quintic threefold V satisfy the identity given in item (d) on page 22.*

Theorem 2.2 *(Katz,Johnsen/Kleiman) For $d \leq 9$, n_d is the number of rational curves of degree d contained in the quintic threefold V.*

Theorem 2.3 *(Pandharipande,Cox/Katz) If the strong form of the Clemens Conjecture for the quintic threefold V holds for $d \leq 10$, then*

$$n_{10} = \#\ \textit{rational curves of degree 10 on } V + 6 \times 17,601,000.$$

In its weakest form, the Clemens Conjecture asserts that for each d, a generic quintic threefold contains only finitely many rational curves of degree d. Currently, this has been proved for $d \leq 9$. So for higher degrees, we need to do two things:

- Prove the Clemens Conjecture.
- Relate n_d to the number of rational curves of degree d on V.

These are both open problems. Theorem 2.3 indicates that the relation in the second bullet may be nontrivial.

3 The Quintic Mirror

Items (a)–(c) page 22 come from the Hodge theory of the *quintic mirror* V°. This is the threefold defined as follows:

- Start with the quintic hypersurfaces in $\mathbb{P}^4$ defined by

$$x_1^5+x_2^5+x_3^5+x_4^5+x_5^5+\psi x_1x_2x_3x_4x_5=0, \tag{3.1}$$

  where $\psi \neq 0$ is a complex number such that $\psi^5 \neq -5^5$.

- Take the quotient of the hypersurface under the action of the group

$$G=\{(a_1,\ldots,a_5)\in\mathbb{Z}_5^5 \mid \textstyle\sum_i a_i \equiv 0 \bmod 5\}/\mathbb{Z}_5,$$

  where the $\mathbb{Z}_5$ is embedded diagonally and $g=(a_1,\ldots,a_5)\in G$ acts on $\mathbb{P}^4$ as

$$g\cdot(x_1,\ldots,x_5)=(\mu^{a_1}x_1,\ldots,\mu^{a_5}x_5).$$

  Here, $\mu=e^{2\pi i/5}$ is a primitive fifth root of unity.

- Finally, V_ψ° is a resolution of singularities of the quotient hypersurface.

This gives a 1-dimensional family V_ψ° of smooth threefolds parametricized by ψ. Furthermore, V_ψ° is a Calabi-Yau threefold, so that $H^{0,3}(V_\psi^\circ)$ has dimension 1. A holomorphic 3-form on V_ψ° is

$$\begin{aligned}\Omega = \mathrm{Res}\Big(\frac{\psi}{x_1^5+\ldots+x_5^5+\psi x_1\cdots x_5}&(x_1dx_2\wedge dx_3\wedge dx_4\wedge dx_5-\\ &x_2dx_1\wedge dx_3\wedge dx_4\wedge dx_5+x_3dx_1\wedge dx_2\wedge dx_4\wedge dx_5-\\ &x_4dx_1\wedge dx_2\wedge dx_3\wedge dx_5+x_5dx_1\wedge dx_2\wedge dx_3\wedge dx_4)\Big).\end{aligned}$$

The periods of Ω, as we vary $x=\psi^{-5}$, satisfy the Picard-Fuchs equation given in item (a) page 22. Thus the functions $y_0(x)$ and $y_1(x)$ in item (b) on page 22 are periods of the quintic mirror family.

Furthermore, using the Gauss-Manin connection ∇, we define the *Yukawa coupling* to be

$$\langle\theta,\theta,\theta\rangle=\int\Omega\wedge\nabla_\theta\nabla_\theta\nabla_\theta\,\Omega \tag{3.2}$$

where $\theta=x\frac{d}{dx}$. This Yukawa coupling satisfies the differential equation in item (c) on page 22.

Now we come to item (d) on page 22, which is the astonishing equation

$$5+\sum_{d=1}^{\infty}n_d d^3\frac{q^d}{1-q^d}=\frac{5}{(1+5^5x)}\,\frac{1}{y_0(x)^2}\left(\frac{q}{x}\frac{dx}{dq}\right)^3.$$

This says that, after a change of variable, the Yukawa coupling on the quintic mirror gives the instanton numbers on the quintic threefold. Hence we have a link between

- the enumerative geometry of the quintic threefold, and
- the Hodge theory of the quintic mirror.

This is one of the most amazing aspects of mirror symmetry.

Finally, as a hint of things to come, let's explain the relation between the quintic threefold and its mirror in more intrinsic terms. First observe that $\omega_{\mathbb{P}^4} = \mathscr{O}_{\mathbb{P}^4}(-5)$ has dual $\mathscr{O}_{\mathbb{P}^4}(5)$. It follows that the quintic threefold V is an anticanonical divisor, which automatically makes it Calabi-Yau. Furthermore, $\mathbb{P}^4$ is the toric variety of the standard 4-simplex

$$\Delta_4 = \mathrm{conv}(0, e_1, e_2, e_3, e_4) \subset M_{\mathbb{R}} = \mathbb{R}^4.$$

While Δ_4 is not reflexive, one can show that

$$\Delta = 5\Delta_4 - (1,1,1,1)$$

is reflexive and satisfies $X_\Delta = \mathbb{P}^4$.

The polar of Δ is the reflexive polytope

$$\Delta^\circ = \mathrm{conv}(-e_1 - e_2 - e_3 - e_4, e_1, e_2, e_3, e_4) \subset N_{\mathbb{R}}.$$

This gives the "dual" toric variety X_{Δ°. Section 4.2 of *Mirror Symmetry and Algebraic Geometry* shows that:

- $X_{\Delta^\circ} = \mathbb{P}^4/G$, where G is the group introduced above.
- The homogeneous coordinate ring of X_{Δ° is $\mathbb{C}[x_0, \ldots, x_4]$. This is graded by $\mathbb{Z} \oplus G$, where the degree of a monomial $x_0^{a_0} \cdots x_4^{a_4}$ is

$$\left(\sum_{i=0}^4 a_i, (-a_1 - a_2 - a_3 a_4, a_1, a_2, a_3, a_4)\right) \in \mathbb{Z} \oplus G.$$

- The anticanonical divisor has degree $(5,0) \in \mathbb{Z} \oplus G$, where $0 \in G$ is the identity. Furthermore, the only monomials of degree $(5,0)$ are

$$x_0^5,\ x_1^5,\ x_2^5,\ x_3^5,\ x_4^5,\ x_0x_1x_2x_3x_4.$$

The last bullet shows that anticanonical hypersurfaces on X_{Δ° are defined by equations of the form

$$a_0x_0^5 + a_1x_1^5 + a_2x_2^5 + a_3x_3^5 + a_4x_4^5 + a_6x_0x_1x_2x_3x_4 = 0.$$

However, using the torus action to rescale the variables individually, one easily sees that every such hypersurface is isomorphic to one defined by an equation of the form (3.1). Hence the construction of the quintic mirror is a special case of the Batyrev mirror construction, which will be described in more detail in Section 5.

4 Superconformal Field Theory

Our next task is to discuss the physics which led Candelas and his coworkers to their formulas for n_d. But before plunging into mirror symmetry, let me point out that modern mathematical physics uses an amazing amount of algebraic geometry and commutative algebra.

For example, consider the following brief description of a *Landau-Ginzburg theory*. In the Lagrangian formulation of such a theory, the most important term of the action is

$$S = \int d^2z d^2\theta F(\Phi_i),$$

where:

- z is a local coordinate on the Riemann surface.
- θ is a fermionic superspace coordinate.
- F is a weighted homogeneous polynomial.
- Φ_i is a chiral superfield.

In this situation, the Euler-Lagrange equations are

$$\frac{\partial F}{\partial \phi_i} = 0,$$

where ϕ_i is the bosonic component of Φ_i.

Starting from the vaccum state V, we can create new states $P(\Phi_i)(V)$, where P is a polynomial in the ϕ_i. By the Euler-Lagrange equations, these states correspond to elements of the quotient

$$\mathbb{C}[\phi_1, \ldots, \phi_n]/\langle \tfrac{\partial F}{\partial \phi_1}, \ldots, \tfrac{\partial F}{\partial \phi_n}\rangle. \tag{3.3}$$

This is the *chiral ring* of the Landau-Ginzburg theory.

Now, if you are like me, most of the above made absolutely no sense. But (3.3) is a Jacobian ring! These are objects I know well—they're even mentioned in my papers! But a chiral ring? What's that?

This is the problem faced by mathematicians trying to understand mirror symmetery—the physics is very sophisticated. For me, the most frustrating aspect is that I don't have access to the intuitions that lie behind these magnificent but mathematically nonrigorous theories.

Keeping this problem in mind, let me say a few words about mirror symmetry. Brace yourself for a lot of incomprehensible words.

Given a smooth Calabi-Yau threefold V and a complexified Kähler class (to be described in Section 6), we get a $N = 2$ heterotic *superconformal field theory* (SCFT) called a σ-model. Such theories deal with strings propagating in $\mathbb{R}^{3,1} \times V$, though we typically ignore the spacetime $\mathbb{R}^{3,1}$ and concentrate on the V part. This leads to maps from Riemann surfaces into V.

Any $N = 2$ SCFT includes a Hilbert space H of states and a representation of $\mathfrak{u}(1) \times \mathfrak{u}(1)$ on H. For the σ model case, this representation has eigenspaces:

$$\begin{aligned} (p,q) - \text{eigenspace} &= H^q(V, \wedge^p T_V) \simeq H^{3-p,q}(V) \\ (-p,q) - \text{eigenspace} &= H^q(V, \Omega^p_V) \simeq H^{p,q}(V). \end{aligned} \tag{3.4}$$

We also note that the moduli space of SCFTs coming from σ-models is governed by

- Complex moduli (vary the complex structure of V).
- Kähler moduli (vary the complexified Kähler class).

In the 1980's, it was noticed that changing the sign of the first generator of the $\mathfrak{u}(1) \times \mathfrak{u}(1)$ representation gave an *abstract* SCFT isomorphic to the original one. The basic idea of mirror symmetry is that *this abstract SCFT should be the σ-model of some other Calabi-Yau threefold, the* **mirror** V° *of* V. Since the sign change interchanges the eigenspaces (3.4), it follows that

$$H^{p,q}(V) \simeq H^{3-p,q}(V^\circ).$$

This also interchanges complex and Kähler moduli, so that the complex (resp. Kähler) moduli of V becomes the Kähler (resp. complex) moduli of V°.

Example 4.1 The quintic threefold V has 1-dimensional Kähler moduli because $H^2(V,\mathbb{C}) = H^{1,1}(V) \simeq \mathbb{C}$. It follows that the quintic mirror V° should have 1-dimensional complex moduli. So the presence of the single moduli parameter $x = \psi^{-5}$ for the quintic mirror is no accident.

The fact that V and V° give isomorphic SCFTs implies that they have the same *threepoint correlation functions*. For the quintic threefold V, the correlation function of interest is $\langle H,H,H\rangle$, where H is the hyperplane section of V. This starts off as some sort of Feynman path integral but can be reduced to

$$\langle H,H,H\rangle = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}.$$

In the right-hand side, $H \cup H \cup H = 5$ is the "topological term" and the n_d are holomorphic instantons which arise as non-perturbative world sheet corrections.

On the mirror side, the Yukawa coupling $\langle \theta,\theta,\theta\rangle$ defined in (3.2) is also a correlation function, though to normalize it, we need to divide by $y_0(x)^2$. Then mirror symmetry says that these correlation functions coincide once we change variables according to the mirror map. This is the map which takes xd/dx to qd/dq, where q is as in item (d) on page 22. Thus we obtain the equation

$$5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = \frac{5}{(1+5^5x)} \frac{1}{y_0(x)^2} \left(\frac{q}{x}\frac{dx}{dq}\right)^3$$

from page 22.

5 The Batyrev Mirror Construction

Now let $\Delta \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional reflexive poltyope with polar $\Delta^\circ \subset N_{\mathbb{R}}$. Then Batyrev's basic observation is each polytope gives a family of anticanonical hypersurfaces

$$\begin{gathered} \overline{V} \subset X_\Delta \\ \overline{V}^\circ \subset X_{\Delta^\circ} \end{gathered}$$

each of which is a Calabi-Yau variety of dimension $n-1$. This is the *Batyrev mirror construction*. Here are some interesting facts about Batyrev mirrors:

- We write $\overline{V}$ and $\overline{V}^\circ$ instead of V and V° is that the former may be singular. In general, one needs to desingularize. However, since a Calabi-Yau variety has trivial canonical class, this needs to be done without changing the canonical class—a so-called "crepant desingularization". This works nicely when $n = 4$ (the case of interest to physics), but in higher dimensions one has to settle for resolutions where V and V° are orbifolds (of an especially nice type).
- When $n = 4$, V and V° are smooth Calabi-Yau threefolds, and it is expected that they should satisfy mirror symmetry. There is a physics "proof" of this when Δ is a reflexive simplex, but the general case is still open.

- In general, when V and V° have dimension $n-1$, we have

$$\dim H^{1,1}(V) = \dim H^{n-2,1}(V^\circ) \quad \text{and} \quad \dim H^{n-2,1}(V) = \dim H^{1,1}(V^\circ).$$

  When V and V°, this takes care of all Hodge numbers. But in higher dimensions,

$$\dim H^{p,q}(V) \neq \dim H^{n-1-p,q}(V^\circ)$$

  can occur in cases when V and V° are simplicial but not smooth. To remedy this, one defines *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}(V)$ which satisfy the mirror equation. These are related to the recently-defined *orbifold cohomology* of an orbifold.

- In 1992, researchers discovered 7555 weighted projective spaces which gave rise to Calabi-Yau threefolds. But this list exhibited only a partial symmetry. How could this be consistent with mirror symmetry? The answer is that the list was produced before Batyrev's definition of reflexive polytope. When the list was recomputed in 1994 and all of the "missing mirrors" were found using Batyrev duality.

6 Other Aspects of Mirror Symmetry

Here are some of the many aspects of mirror symmetry not mentioned so far:

- **Complexified Kähler Cone.** Let V be a Calabi-Yau of dimension ≥ 3. Then

$$H^2(V,\mathbb{R}) = H^{1,1}(V)_{\mathbb{R}} \supset K \to \text{Kähler cone of } V.$$

  The complexified Kähler space is

$$K_{\mathbb{C}} = \{B+iJ \mid J \in K, B \in H^2(V,\mathbb{R})\}/H^2(V.\mathbb{Z}).$$

  This is a basic building block for the Kähler moduli space of V.

- **Boundary Points of Moduli Spaces.** Mathematical versions of mirror symmetry take place at boundary points of moduli spaces. For the quintic threefold, $x = \psi^{-5}$ is a local coordinate for a *maximally unipotent boundary point*. The corresponding point on the boundary of the Kähler moduli space is a *large radius point*.

- **Multiple Mirrors and the GKZ Decompostion.** When we compare the complex moduli of V to the Kähler moduli of V°, the latter is typically much smaller than the former. This doesn't seem consistent with mirror symmetry. The answer is to enlarge the Kähler moduli space. For example, in the toric case, it can happen that different simplicial fans Σ can refine the normal fan of a reflexive polytope Δ. The toric varieties X_Σ have the same H^2 but different Kähler cones. All of this can be described torically using the *GKZ decomposition*. (This is the smaller version of the decomposition, which in physics terms corresponds to the σ-models of the Calabi-Yau hypersurfaces in the X_Σ. The larger version includes more cones which correspond to certain Landau-Ginzburg theories.)

- **Mirror Theorems.** For Calabi-Yau complete intersections in Fano toric varieties, Givental and Lian/Liu/Yau have proved very powerful Mirror Theorems which generalize the formula at the bottom of page 22.

- **Quantum Cohomology and the A-Variation of Hodge Structure.** Gromov-Witten invariants can be used to define a deformation of cup product called *quantum cohomology*. This is related to a (1+1)-dimensional *topological quantum field theory*. One can also turn this into the *A-variation of Hodge structure,* which is a polarized variation of Hodge structure over the Kähler moduli space. Mirror symmetry can be formulated as the assertion that the A-VHS of V over its Kähler moduli space is isomorphic (as a polarized VHS) to the geometric VHS of V° over its complex moduli space.
- **Conifold Transitions.** Every family of Calabi-Yau threefolds gives a model of the universe. In 1995, Greene/Morrison/Strominger discovered how to get from one model to another by what is called a *conifold transition*. This can be pictured as follows:

$$\begin{array}{ccccc}
\widetilde{V} & \xrightarrow[\text{degenerate}]{\qquad} & \overline{V} & \xleftarrow[\text{resolve}]{\qquad} & V \\
\downarrow & \downarrow & \downarrow & & \downarrow \\
\begin{array}{c}\text{black}\\ \text{hole}\end{array} & \begin{array}{c}\text{vanishing}\\ \text{cycle}\end{array} & \text{massless} & & \begin{array}{c}\text{elementary}\\ \text{particle}\end{array}
\end{array}$$

- **Modern Mirror Symmetry.** What I have described so far can be described as "classical" mirror symmetry. More recently, people have explored the following topics:
  - ◇ Homological mirror symmetry (Kontsevich).
  - ◇ D-branes, F-theory, M-theory (physics).
  - ◇ V° is the complexified moduli space of Lagrangian tori on V (Strominger/Yau/Zaslow).
  - ◇ Vertex algebras (Borisov).
  - ◇ Chiral De Rham complexes (Malikov/Schechtman)
- **Science Fiction.** In 1998, Stephen Baxter wrote the science fiction novel *Moonseed.* Here is some of the dialog between characters named Monica and Alfred:

  ****

  Monica: Now, the 6 missing dimensions are there, but they are crumpled up … The trouble is, there are tens of thousands of ways for space to crumple up … And in each internal space, the strings adopt a different solution.

  ****

  Monica: Theoreticians are suggesting there is a—tear in space—at the heart of Venus.

  Alfred: A tear?

  Monica: A way into another internal space. Exotic particles as massive as bacteria.

  ****

  Alfred (in an email to Monica): Take one of your 10-dimensional string objects … As you approach zero width, you generate quantum-mechanical waves … The waves are extremal black holes.

Acknowledgements

The author would like to thank Alicia Dickenstein for inviting me to give this minicourse at the University of Buenos Aires.

In preparing these lectures, I used the following sources:

- Lecture I is based in part on material from my paper *Toric varieties and toric resolutions* listed in the Bibliography.
- Section 2 of Lecture II is based on Section 5.6 of Fulton's book listed in the Bibliography.
- Section 3 of Lecture II is based on a lecture given by Michel Brion at the Summer School on Toric Varieties held in Grenoble in July 2000.
- Section 5 of Lecture II and most of Lecture III is based in part on my book *Mirror Symmetry and Algebraic Geometry* written with Sheldon Katz.

Bibliography

Basic References on Toric Varieties

1. J.-L. Brylinski, *Eventails et variétés toriques*, in *Séminaire de les Singularités de Surfaces* (ed. by M. Demazure, H. Pinkham and B. Tessier), Lecture Notes in Math. **777**, Springer-Verlag, New York-Berlin-Heidelberg, 1980, 247–288.
2. V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.
3. G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Springer-Verlag, New York-Berlin-Heidelberg, 1996.
4. W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton, NJ, 1993.
5. T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to Toric Varieties*, Springer-Verlag, New York-Berlin-Heidelberg, 1988.

Basic References on Algebraic Geometry

1. D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Second Edition, Springer-Verlag, New York-Berlin-Heidelberg, 1997.
2. D. Eisenbud and J. Harris, *The Geometry of Schemes*, Springer-Verlag, New York-Berlin-Heidelberg, 2000.
3. J. Harris, *Algebraic Geometry: A First Course*, Springer-Verlag, New York-Berlin-Heidelberg, 1992.
4. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Berlin-Heidelberg, 1977.
5. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press, Cambridge, 1988.
6. I. R. Shafarevich, *Basic Algebraic Geometry*, Second Revised and Expanded Edition, Volumes 1 and 2, Springer-Verlag, New York-Berlin-Heidelberg, 1994.

Basic References on Cones and Polytopes

1. A. Brønsted, *An Introduction to Convex Polytopes*, Springer-Verlag, New York-Berlin-Heidelberg, 1983.
2. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.
3. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
4. G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York-Berlin-Heidelberg, 1995.

Introductory Articles on Toric Varieties

1. D. Cox, *Toric varieties and toric resolutions*, in *Resolution of Singularities* (H. Hauser, J. Lipman, F. Oort and A. Quirós, eds.), Birkhäuser, Basel-Boston-Berlin, 2000, 259–284.
2. M. Reid, *Review of "Convex Bodies and Algebraic Geometry: An Introduction to Toric Varieties" by T. Oda*, Bull. AMS **21** (1989), 360–364.

Survey Articles on Toric Varieties

1. D. Cox, *Recent developments in toric geometry*, in *Algebraic Geometry—Santa Cruz 1995* (J. Kollár, R. Lazarsfeld and D. Morrison, eds.), Volume 2, AMS, Providence, RI, 1997, 389–436.
2. T. Oda, *Geometry of toric varieties*, in *Proceedings of Hyderabad Conference on Algebraic Groups* (S. Ramanan, ed.), Manoj Prakashan, Madras, 1991, 407–440.
3. T. Oda, *Recent topics on toric varieties*, AMS Translations **172** (1996), 77–91.