Toric Tutorial

Schedule of Lectures:

- Lecture I: 9-10
  What is a Toric Variety?
  David Cox

- Lecture II: 10:10–11:10
  Toric Ideals, Real Toric Varieties,
  the Moment Map, etc.
  Frank Sottile

- Lecture III: 11:15–12:15
  Lattice Points, Mixed Subdivisions,
  the BKK Theorem, etc.
  Maurice Rojas

Outline of Lecture I:

1. Varieties
2. Toric Varieties
3. Examples
4. Cones
5. Cones and Affine Toric Varieties
6. Normality
7. Fans and Toric Varieties
8. Properties of Toric Varieties
9. Homogeneous Coordinates
10. The Toric Variety of a Polytope
11. Polytopes and Homogeneous Coordinates

1. Varieties

Most common varieties over $\mathbf{C}$:

- $\mathbf{C}^n$ and *affine varieties*

$$V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbf{C}^n$$

- $\mathbf{P}^n$ and *projective varieties*

$$V = \mathbf{V}(F_1, \ldots, F_s) \subset \mathbf{P}^n$$

Example 1.1. Let $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Then $(\mathbf{C}^*)^n \subset \mathbf{C}^n$ is an affine variety via

$$(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 1/t_1 \cdots t_n)$$
$$(\mathbf{C}^*)^n \simeq \mathbf{V}(x_1 x_2 \cdots x_{n+1} - 1) \subset \mathbf{C}^{n+1}.$$

$(\mathbf{C}^*)^n$ is the n-*dimensional complex torus* and is the "toric" in "toric variety".

$V \setminus W$ is *Zariski open* in V when $W \subset V$.

Example 1.2. $(\mathbf{C}^*)^n = \mathbf{C}^n \setminus \mathbf{V}(x_1 \cdots x_n)$ is Zariski open in $\mathbf{C}^n$.

V is *irreducible* if it can't be written $V = V_1 \cup V_2$ for $V_1 \neq V$ and $V_2 \neq V$.

2. Toric Varieties

Definition 2.1. A *toric variety* V is irreducible, contains $(\mathbf{C}^*)^n$ as a Zariski open subset, and the action of $(\mathbf{C}^*)^n$ on itself extends to an action on V.

Example 2.2. $\mathbf{C}^n$ and $\mathbf{P}^n$ are toric varieties, where $(\mathbf{C}^*)^n \subset \mathbf{P}^n$ via

$$(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n).$$

Also:

1. $m \in M = \mathbf{Z}^n$ gives the *character* $\chi^m : (\mathbf{C}^*)^n \to \mathbf{C}^*$ defined by

$$\chi^m(t_1, \ldots, t_n) = t_1^{m_1} \cdots t_n^{m_n}.$$

χ^m is a *Laurent monomial*.

2. $u \in N = \mathbf{Z}^n$ gives the *1-parameter subgroup* $\lambda^u : \mathbf{C}^* \to (\mathbf{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{u_1}, \ldots, t^{u_n}).$$

3. $m \in M, u \in N$ give $\langle m, u \rangle = m \cdot u \in \mathbf{Z}$.

3. Examples

Example 3.1. V, W toric $\Rightarrow$ so is $V \times W$.

Example 3.2. $\mathbf{V}(y^2 - x^3) \subset \mathbf{C}^2$ is a toric variety via $t \mapsto (t^2, t^3)$. This is *non-normal*. The only 1-dimensional normal toric varieties are $\mathbf{C}^*$, $\mathbf{C}$ and $\mathbf{P}^1$.

Example 3.3. $V = \mathbf{V}(xy - zw) \subset \mathbf{C}^4$ is a toric variety via

$$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}).$$

Since $xy = zw$ on V, we have

$$x^a y^b z^c = x^a y^b \Big(\frac{xy}{w}\Big)^c = x^{a+c} y^{b+c} w^{-c}.$$

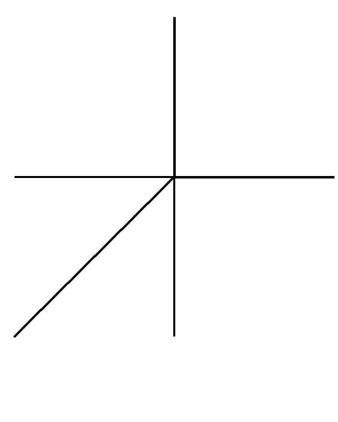
So $\chi^m = \chi^{(a,b,c)}$ extends to V if

$$a \geq 0,\ b \geq 0,\ a + c \geq 0,\ b + c \geq 0.$$

Example 3.4. For $\mathbf{P}^2$, $u \in N$ gives a 1-parameter subgroup $\lambda^u : \mathbf{C}^* \to \mathbf{P}^2$. What is $\lim_{t\to 0} \lambda^u(t)$? Let $u = (a,b) \in N = \mathbf{Z}^2$, so that $\lambda^u(t) = (1, t^a, t^b)$. Then:

$$\lim_{t\to 0} \lambda^u(t) = \begin{cases} (1,0,0) & a,b>0 \\ (1,0,1) & a>0, b=0 \\ (1,1,0) & a=0, b>0 \\ (1,1,1) & a=b=0 \\ (0,0,1) & a>b, b<0 \\ (0,1,0) & a<0, a<b \\ (0,1,1) & a<0, a=b. \end{cases}$$

To see how the fifth case works, note $\lim_{t\to 0}(1, t^a, t^b) = \lim_{t\to 0}(t^{-b}, t^{a-b}, 1)$. This gives the picture:



4. Cones

Let $N_{\mathbf{R}} = \mathbf{R}^n$. A *rational polyhedral cone* $\sigma \subset N_{\mathbf{R}}$ is:

$$\sigma = \left\{\lambda_1 u_1 + \cdots + \lambda_\ell u_\ell \mid \lambda_1, \ldots, \lambda_\ell \geq 0\right\},$$

where $u_1, \ldots, u_\ell \in N = \mathbf{Z}^n$. Then:

- σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.
- $\dim \sigma$ is the *dimension* of σ.
- A *face* of σ is $\{\ell = 0\} \cap \sigma$, where ℓ is a linear form which is ≥ 0 on σ.
- An *edges* ρ of σ is a 1-dim face.
- The *primitive element* n_ρ is the unique minimal generator of $\rho \cap N$.
- σ is generated by the n_ρ of its edges.
- A *facet* of σ is a codimension-1 face.

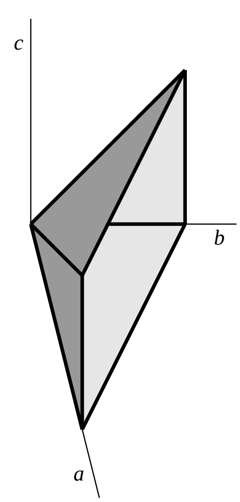
Definition 4.1. A strongly convex rational polyhedral σ has *dual cone*

$$\sigma^\vee = \{m \in M_{\mathbf{R}} = \mathbf{R}^n \mid \langle m, u\rangle \geq 0\ \forall u \in \sigma\}.$$

This is rational polyhedral of dim n.

Elements of N are called *lattice points* of $N_{\mathbf{R}}$ and elements of M are called *lattice points* of $M_{\mathbf{R}}$.

Example 4.2. Consider $\sigma \subset N_{\mathbf{R}} = \mathbf{R}^3$:



This cone has primitive elements

$$n_1 = (1, 0, 0),\ n_2 = (0, 1, 0),$$
$$n_3 = (1, 0, 1),\ n_4 = (0, 1, 1)$$

and inward pointing normals

$$m_1 = (1, 0, 0),\ m_2 = (0, 1, 0),$$
$$m_3 = (0, 0, 1),\ m_4 = (1, 1, -1).$$

These generate the dual cone $\sigma^\vee$ in $M_{\mathbf{R}}$. Thus $(a, b, c) \in \sigma^\vee$ iff

$$a \geq 0,\ b \geq 0,\ a + c \geq 0,\ b + c \geq 0.$$

In general, the set of linear combinations of characters χ^m for $m \in \sigma^\vee \cap M$ is

$$\mathbf{C}[\sigma^\vee \cap M].$$

This is a ring since $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

In terms of Laurent monomials, we have

$$\mathbf{C}[\sigma^\vee \cap M] \subset \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

5. Cones and Affine Toric Varieties

A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbf{R}}$ determines the affine toric variety U_σ as follows.

By *Gordan's Lemma*, $\sigma^\vee \cap M$ is generated over $\mathbf{Z}_{\geq 0}$ by $m_1, \ldots, m_\ell \in M$. Map $(\mathbf{C}^*)^n \to \mathbf{C}^\ell$ by sending $(t_1, \ldots, t_n)$ to

$$\big(\chi^{m_1}(t_1, \ldots, t_n), \ldots, \chi^{m_\ell}(t_1, \ldots, t_n)\big).$$

Then $U_\sigma \subset \mathbf{C}^\ell$ is the *Zariski closure* of the image of this map.

We can think of this is as follows. Let $y_1, \ldots, y_\ell$ be variables, and consider

$$\mathbf{C}[y_1, \ldots, y_\ell] \to \mathbf{C}[\sigma^\vee \cap M] = \mathbf{C}[\chi^{m_1}, \ldots, \chi^{m_\ell}]$$

defined by sending y_i to χ^{m_i}. This map is onto and its kernel $I \subset \mathbf{C}[y_1, \ldots, y_\ell]$ consists of all algebraic relations among the χ^{m_i}. If $I = \langle f_1, \ldots, f_s \rangle$, then

$$U_\sigma = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbf{C}^\ell.$$

Examples 5.1 and 5.3. For the cone of Example 4.2, the inward normals

$$m_1 = (1, 0, 0),\ m_2 = (0, 1, 0),$$
$$m_3 = (0, 0, 1),\ m_4 = (1, 1, -1)$$

generate $\sigma^\vee \cap M$.

Thus $\mathbf{C}[\sigma^\vee \cap M] = \mathbf{C}[\chi^{m_1}, \chi^{m_2}, \chi^{m_3}, \chi^{m_4}]$. Then $m_1 + m_2 = m_3 + m_4$ implies that $xy - zw$ is in the kernel of

$$\mathbf{C}[x, y, z, w] \to \mathbf{C}[\sigma^\vee \cap M].$$

In fact, $xy - zw$ generates the kernel, so

$$U_\sigma = \mathbf{V}(xy - zw) \subset \mathbf{C}^4.$$

This is the toric variety of Example 3.3.

In general, $\mathbf{C}[\sigma^\vee \cap M]$ is the *coordinate ring* of U_σ. This is the ring of polynomial functions on the affine variety U_σ.

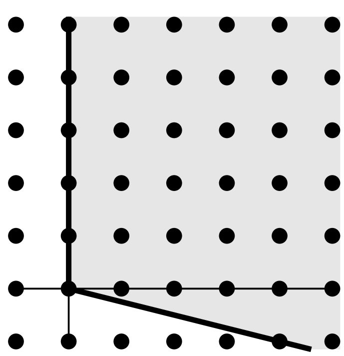
Thus $\mathbf{C}[\sigma^\vee \cap M]$ tells us which characters on $(\mathbf{C}^*)^n$ extend to functions defined on all of U_σ.

6. Normality

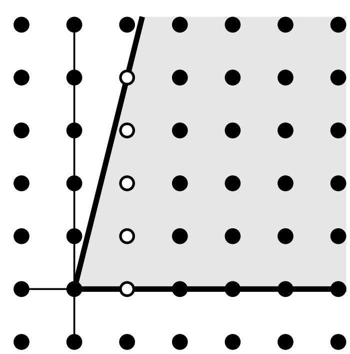
A variety is *normal* if its local rings are integrally closed in their fields of fractions. The affine variety U_σ is normal.

Question: When is an affine toric variety normal?

Example 6.1. Consider



The cone σ



The cone $\sigma^\vee$

The generators of $\sigma^\vee \cap M$ are $m_i = (1, i)$ for $i = 0, \ldots, 4$.

$U_\sigma \subset \mathbf{C}^5$ is the Zariski closure of the image of $(\mathbf{C}^*)^2 \to \mathbf{C}^5$ defined by

$$(t,u) \mapsto (t, tu, tu^2, tu^3, tu^4).$$

What if we omit some of the m_i?

1. m_0, m_4 give $(\mathbf{C}^*)^2 \to \mathbf{C}^2$ where

$$(t,u) \mapsto (t, tu^4).$$

The Zariski closure is $\mathbf{C}^2$ but the map is 4-to-1 and m_0, m_4 don't generate $\mathbf{Z}^2$. Messed up the lattice.

2. m_0, m_1, m_4 give $(\mathbf{C}^*)^2 \to \mathbf{C}^3$ where

$$(t,u) \mapsto (t, tu, tu^4).$$

The Zariski closure is $x^3 z = y^4$. The map is 1-to-1 and m_0, m_1, m_4 generate $\mathbf{Z}^2$. Not normal since codim(sing) $= 1$.

Let $\sigma \subset N_{\mathbf{R}} = \mathbf{R}^n$ be a strongly convex rational polyhedral cone. Given $m_i \in \sigma^\vee \cap M$ for $i = 1, \ldots, \ell$, the χ^{m_i} give

$$(\mathbf{C}^*)^n \longrightarrow \mathbf{C}^\ell.$$

Theorem 6.2. The Zariski closure of the image of this map is the normal affine toric variety U_σ determined by σ and N if and only if $\sigma^\vee \cap M$ is generated over $\mathbf{Z}_{\geq 0}$ by m_i for $i = 1, \ldots, \ell$.

This shows that an affine toric variety is normal precisely when you use *all* lattice points in the dual cone.

7. Fans and Toric Varieties

A *fan* is a finite collection Σ of cones in $N_{\mathbf{R}}$ with the properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

Each $\sigma \in \Sigma$ gives the affine toric variety U_σ, and if τ is a face of σ, then U_τ is a Zariski open subset of U_σ.

Definition 7.1. Given a fan Σ in $N_{\mathbf{R}}$, X_Σ is the variety obtained from the affine varieties U_σ, $\sigma \in \Sigma$, by gluing together U_σ and U_τ along their common open subset $U_{\sigma \cap \tau}$ for all $\sigma, \tau \in \Sigma$.

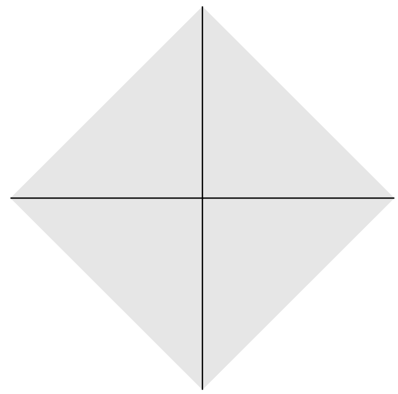
Example 7.2. For $\sigma \subset N_{\mathbf{R}}$, we get a fan by taking faces of σ (including σ). The toric variety of this fan is U_σ.

Example 7.3. The fan for $\mathbf{P}^1$ is:

The cones $\sigma_1 = [0, \infty)$ and $\sigma_2 = (-\infty, 0]$ give U_1 with coordinate ring $\mathbf{C}[t]$ and U_2 with coordinate ring $\mathbf{C}[t^{-1}]$, which patch in the usual way to give $\mathbf{P}^1$.

Example 7.4. For a basis $e_1, \ldots, e_n$ of $N = \mathbf{Z}^n$, set $e_0 = -e_1 - \cdots - e_n$. Then $\mathbf{P}^n$ is the toric variety of the fan whose cones are generated by all proper subsets of $\{e_0, e_1, \ldots, e_n\}$. When $n = 2$, this fan appeared in Example 3.4.

Example 7.5. The fan for $\mathbf{P}^1 \times \mathbf{P}^1$ is:



8. Properties of Toric Varieties

There are one-to-one correspondences between the following:

- The limits $\lim_{t\to 0} \lambda^u(t)$ for $u \in |\Sigma| = \cup_{\sigma\in\Sigma}\, \sigma$ ($|\Sigma|$ is the *support* of Σ).
- The cones $\sigma \in \Sigma$.
- The orbits of the torus action on X_Σ.

The correspondences is as follows:

An orbit corresponds to a cone σ iff $\lim_{t\to 0} \lambda^u(t)$ exists and lies in the orbit for all u in the relative interior of σ.

For an orbit $\mathrm{orb}(\sigma)$, we have:

- $\dim \sigma + \dim \mathrm{orb}(\sigma) = n$.
- $\mathrm{orb}(\sigma) \subset \overline{\mathrm{orb}(\tau)}$ if and only if $\tau \subset \sigma$.

Theorem 8.1. Let X_Σ be the toric variety of a fan Σ in $N_{\mathbf{R}}$. Then:

- X_Σ is compact $\Leftrightarrow$ $|\Sigma| = N_{\mathbf{R}}$.
- X_Σ is smooth $\Leftrightarrow$ all $\sigma \in \Sigma$ are smooth (generated by a subset of a $\mathbf{Z}$-basis).
- X_Σ is simplicial (has finite quotient singularities) $\Leftrightarrow$ all Σ are simplicial (generated by a subset of a $\mathbf{Q}$-basis).

9. Homogeneous Coordinates

Assign a variable to each 1-dimensional cone in the fan of X_Σ. Thus:

$\rho_1, \ldots, \rho_r$	1-dim cones
$n_1, \ldots, n_r$	primitive generators
$D_1, \ldots, D_r$	orbit closures in X_Σ
$x_1, \ldots, x_r$	variables.

A monomial $\Pi_i x_i^{a_i}$ gives a divisor $D = \sum_i a_i D_i$, so we write $x^D = \Pi_i x_i^{a_i}$. Given $x^E = \Pi_i x_i^{b_i}$, define $\deg(x^D) = \deg(x^E)$
$\iff D = E + \mathrm{div}(\chi^m)$ for some $m \in M$
$\iff a_i = b_i + \langle n_i, m\rangle$ for some $m \in M$.
This uses $\mathrm{div}(\chi^m) = \sum_i \langle n_i, m\rangle D_i$

deg(x^D) lies in the *Chow group*

$$A_{n-1}(X_\Sigma) = \mathbf{Z}^r/\alpha(M),$$

where $\alpha : M \to \mathbf{Z}^r$ is defined by

$$\alpha(m) = \big(\langle n_1, m\rangle, \ldots, \langle n_r, m\rangle\big).$$

Then $\mathbf{C}[x_1, \ldots, x_r]$ is the *homogeneous coordinate ring* of X_Σ.

Example 9.1. For $\mathbf{P}^n$, we get the ring $\mathbf{C}[x_0, \ldots, x_n]$ with the usual grading.

Example 9.2. For $\mathbf{P}^1 \times \mathbf{P}^1$, we get divisors D_1, D_2 corresponding to the horizontal rays in the fan and divisors D_3, D_4 corresponding to vertical ones. Let the corresponding variables be x_1, x_2, x_3, x_4.

To grade this, define $\mathbf{Z}^4 \to \mathbf{Z}^2$ by

$$(a_1, a_2, a_3, a_4) \mapsto (a_1 + a_2, a_3 + a_4).$$

The kernel of this map is the image of α. Hence

$$\deg(x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}) = (a_1 + a_2, a_3 + a_4).$$

This is precisely the usual bigrading on $\mathbf{C}[x_1, x_2; x_3, x_4]$, where each graded piece consists of bihomogeneous polynomials in x_1, x_2 and x_3, x_4.

To get coordinates, we need an analog of the "irrelevant" ideal $\langle x_0, \ldots, x_n \rangle$ for $\mathbf{P}^n$. We do this as follows.

Given $\sigma \in \Sigma$, set

$$x^{\widehat{\sigma}} = \Pi_{n_i \notin \sigma} x_i$$

and $B = \langle x^{\widehat{\sigma}} \mid \sigma \in \Sigma \rangle$. This uses *all* cones of the fan, while the homogeneous coordinate ring uses only the 1-dim ones.

Also set $G = \mathrm{Hom}_{\mathbf{Z}}(A_{n-1}(X_\Sigma), \mathbf{C}^*)$. This is the kernel of the dual of the map $(\mathbf{C}^*)^r \to (\mathbf{C}^*)^n$ induced by α.

$G \subset (\mathbf{C}^*)^r$ implies that G acts naturally on $\mathbf{C}^r$ and leaves $\mathbf{V}(B)$ invariant. Thus we can form the quotient

$$\big(\mathbf{C}^r \setminus \mathbf{V}(B)\big)/G.$$

Theorem 9.3. Let X_Σ be toric variety where $n_1, \ldots, n_\ell$ span $N_{\mathbf{R}}$. Then:

1. X_Σ is the universal categorical quotient $\big(\mathbf{C}^r \setminus \mathbf{V}(B)\big)/G$.
2. $\big(\mathbf{C}^r \setminus \mathbf{V}(B)\big)/G$ is a geometric quotient if and only if X_Σ is simplicial.

We have $(\mathbf{C}^*)^n \simeq (\mathbf{C}^*)^r/G$ by definition. Then $(\mathbf{C}^*)^r \subset \mathbf{C}^r$ induces

$$(\mathbf{C}^*)^n \simeq (\mathbf{C}^*)^r/G \subset \big(\mathbf{C}^r \setminus \mathbf{V}(B)\big)/G \simeq X_\Sigma.$$

Since $(\mathbf{C}^*)^r$ acts on $\mathbf{C}^r \setminus \mathbf{V}(B)$, $(\mathbf{C}^*)^n$ acts on X_Σ. And categorical quotients preserve normality, so that the quotient is a normal toric variety.

Example 9.4. For $\mathbf{P}^n$, we get the usual representation $\mathbf{P}^n \simeq (\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^*$.

Example 9.5. For $\mathbf{P}^1 \times \mathbf{P}^1$, we have $B = \langle x_1x_3, x_1x_4, x_2x_3, x_2x_4 \rangle$. Then

$$\mathbf{V}(B) = (\{0\} \times \mathbf{C}^2) \cup (\mathbf{C}^2 \times \{0\})$$

and $G \simeq (\mathbf{C}^*)^2$ acts on $\mathbf{C}^4$ via

$$(\lambda, \mu)\cdot(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \mu x_3, \mu x_4).$$

Hence the quotient of Theorem 9.3 is

$$\big((\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^*\big) \times \big((\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^*\big),$$

which is how one represents $\mathbf{P}^1 \times \mathbf{P}^1$ as a quotient.

Example 9.7. (Simplicial, Not Smooth) Let $\sigma \subset N_{\mathbf{R}} = \mathbf{R}^2$ be generated by $n_1 = (1,0)$, $n_2 = (1,2)$. The homogeneous coordinate ring is $\mathbf{C}[x_1, x_2]$, where x_1, x_2 have degree 1 mod 2. Furthermore:

1. $U_\sigma = V(xz - y^2) \subset \mathbf{C}^3$.

2. G acts on $\mathbf{C}^2$ by multiplication by ± 1.

3. The ring of invariants is $\mathbf{C}[x_1, x_2]^G = \mathbf{C}[x_1^2, x_1 x_2, x_2^2]$.

4. The quotient $\pi : \mathbf{C}^2 \to U_\sigma$ is the map $(x_1, x_2) \to (x_1^2, x_1 x_2, x_2^2)$.

Note that $\mathbf{C}^2 \to U_\sigma$ is 2-to-1. This is a a finite quotient singularity.

Example 9.8. (Not Simplicial)

Let σ be the 3-dim cone of Example 3.3. The ring $\mathbf{C}[x_1, x_2, x_3, x_4]$ is graded by $\mathbf{Z}$, where the variables have degrees

$$\begin{aligned} \deg(x_1) &= \deg(x_4) = 1 \\ \deg(x_2) &= \deg(x_3) = -1. \end{aligned}$$

Furthermore:

1. $U_\sigma = \mathbf{V}(xy - zw) \subset \mathbf{C}^4$.

2. $G = \mathbf{C}^*$ acts via $\lambda \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda^{-1} x_2, \lambda^{-1} x_3, \lambda x_4)$.

3. The invariant ring is $\mathbf{C}[x_1, x_2, x_3, x_4]^G = \mathbf{C}[x_1x_2, x_3x_4, x_1x_3, x_2x_4]$.

4. The quotient map $\pi : \mathbf{C}^2 \to U_\sigma$ is $(x_1, x_2, x_3, x_4) \mapsto (x_1x_2, x_3x_4, x_1x_3, x_2x_4)$.

If $p \in U_\sigma$, then:

- $p \neq 0 \Rightarrow \pi^{-1}(p)$ is a G-orbit.
- $p = 0 \Rightarrow \pi^{-1}(p) = (\mathbf{C} \times \{0\} \times \{0\} \times \mathbf{C}) \cup (\{0\} \times \mathbf{C} \times \mathbf{C} \times \{0\})$.

In general, a "categorical quotient" is constructed using the ring of invariants under the group action.

10. The Toric Variety of a Polytope

A *lattice polytope* Δ in $M_{\mathbf{R}} = \mathbf{R}^n$ is the convex hull of a finite subset of M. We represent Δ as an intersection of halfspaces as follows.

For each facet F of Δ, there is an inward normal primitive vector $n_F \in N$ and $a_F \in \mathbf{Z}$ such that

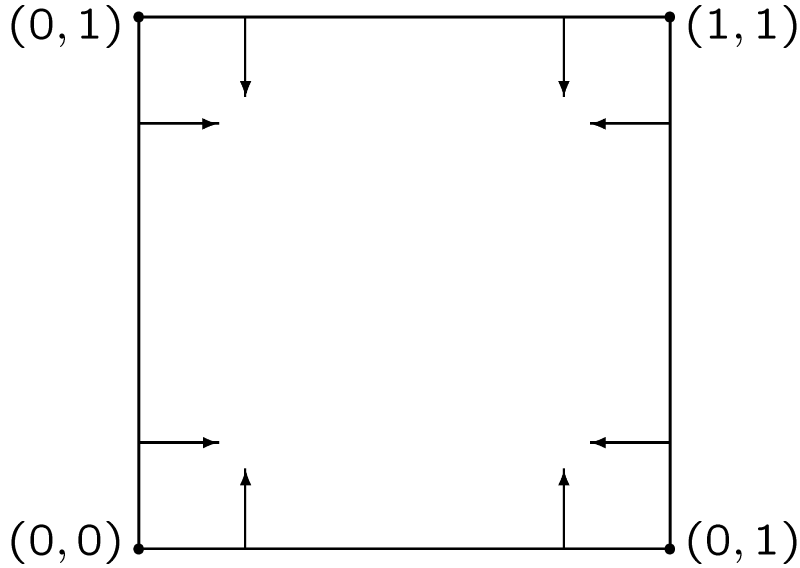
$$\Delta = \bigcap_{F \text{ is a facet}} \{m \in M_{\mathbf{R}} \mid \langle m, n_F \rangle \geq -a_F\}.$$

Given a face $\mathcal{F}$ of Δ, we let $\sigma_{\mathcal{F}}$ be the cone generated by n_F for all facets F containing $\mathcal{F}$. Then

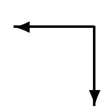
$$\Sigma_\Delta = \{\sigma_{\mathcal{F}} \mid \mathcal{F} \text{ is a face of } \Delta\}$$

is the *normal fan* of Δ. This gives a toric variety X_Δ.

Example 10.1. Consider the unit square in $M_{\mathbf{R}} = \mathbf{R}^2$. The inward normals (not to scale) are:



The four vertices give four 2-dim cones in the normal fan. For example, the vertex $(1, 1)$ gives the 2-dim cone



From here, it is easy to see that we get the fan of Example 7.5. Thus the toric variety is $\mathbf{P}^1 \times \mathbf{P}^1$.

Theorem 10.2. The normal toric variety of a fan Σ in $N_{\mathbf{R}} \simeq \mathbf{R}^n$ is projective if and only if Σ is the normal fan of an n-dimensional lattice polytope in $M_{\mathbf{R}}$.

We also have a 1-to-1 correspondence

$$\sigma_{\mathcal{F}} \in \Sigma_\Delta \longleftrightarrow \mathcal{F} \subset \Delta$$

between cones and faces such that

$$\dim \sigma_{\mathcal{F}} + \dim \mathcal{F} = n.$$

Let $m_1, \ldots, m_\ell$ be the lattice points of Δ. Then sending $(t_1, \ldots, t_n) \in (\mathbf{C}^*)^n$ to

$$\big(\chi^{m_1}(t_1, \ldots, t_n), \ldots, \chi^{m_\ell}(t_1, \ldots, t_n)\big) \in \mathbf{P}^{\ell-1}$$

extends to $X_\Delta \to \mathbf{P}^{\ell-1}$. When $\nu \gg 0$, this map for $\nu\Delta$ is an embedding.

11. Polytopes and Homogeneous Coordinates

Fix a lattice polytope $\Delta \subset M_{\mathbf{R}} = \mathbf{R}^n$. Since 1-dimensional cones of the normal fan correspond to facets of Δ, we get:

$\rho_1, \ldots, \rho_r$	1-dim cones of normal fan
$F_1, \ldots, F_r$	facets of lattice polytope
$x_1, \ldots, x_r$	facet variables.

Given a vertex v, the *vertex monomial* $x^{\hat{v}}$ is the product of variables whose facets miss the v. These generate B, so that $\mathbf{C}^r \setminus \mathbf{V}(B)$ consists of points where at least one vertex monomial is nonzero.

Δ gives some interesting monomials in the coordinate ring. Let

$$\Delta = \bigcap_i \{m \in M_{\mathbf{R}} \mid \langle m, n_i \rangle \geq a_i\}$$

and let $D = \sum_i a_i D_i$. If $m \in \Delta \cap M$, then

$$\mathbf{x}^m = \prod_i x_i^{\langle m, n_i \rangle + a_i}$$

is the Δ-*homogenization* of χ^m. For any monomial x^E, $\deg(x^E) = \deg(x^D)$ iff

$$x^E = \mathbf{x}^m \text{ for some } m \in \Delta \cap M.$$

This gives a 1-to-1 correspondence between monomials of degree $\deg(x^D)$ and lattice points of Δ.

Now consider the map

$$\mathbf{x} = (x_1, \ldots, x_r) \to (\mathbf{x}^{m_1}, \ldots, \mathbf{x}^{m_\ell}).$$

This map has two properties:

- $\mathbf{x} \notin \mathbf{V}(B)$ implies $\mathbf{x}^{m_i} \neq 0$ for some i.
- Recall $G \subset (\mathbf{C}^*)^r$, so $\mu \in G$ gives $\mu\mathbf{x} = (\mu_1 x_1, \ldots, \mu_r x_r)$. Then for each $m_i \in \Delta \cap M$,

$$(\mu\mathbf{x})^{m_i} = \mu_\Delta\, \mathbf{x}^{m_i},$$

where $\mu_\Delta = \mu_1^{a_1} \cdots \mu_r^{a_r}$.

It follows that we get well-defined map

$$X_\Delta = \big(\mathbf{C}^r \setminus \mathbf{V}(B)\big)/G \longrightarrow \mathbf{P}^{\ell-1}.$$

If one restricts this map to $(\mathbf{C}^*)^n \subset X_\Delta$, the result is *exactly* the map given at the end of Section 10. Using $(n-1)\Delta$ instead of Δ gives an embedding.