Topics in Curve and Surface Implicitization

David A. Cox

Amherst College
Outline

**Curves:**

- Moving Lines & $\mu$-Bases
- Moving Curve Ideal & the Rees Algebra
- Adjoint Curves
Outline

**Curves:**
- Moving Lines & $\mu$-Bases
- Moving Curve Ideal & the Rees Algebra
- Adjoint Curves

**Surfaces:**
- Parametrized Surfaces
- Moving Planes & Syzygies
- Affine, Projective & Bihomogeneous
- The Resultant of a $\mu$-Basis
Curve Implicitization

Turn a *parametrization* into an *equation*.

- **Affine**: Turn

\[
x = \frac{a(t)}{c(t)}, \quad y = \frac{b(t)}{c(t)}
\]

into \( F(x, y) = 0 \).
Curve Implicitization

Turn a *parametrization* into an *equation*.

■ **Affine**: Turn

\[ x = \frac{a(t)}{c(t)}, \quad y = \frac{b(t)}{c(t)} \]

into \( F(x, y) = 0 \).

■ **Projective**: Turn

\[ x = a(s, t), \quad y = b(s, t), \quad z = c(s, t) \]

into \( F(x, y, z) = 0 \).
Moving Lines

A *moving line* is an equation

\[ A(s, t)x + B(s, t)y + C(s, t)z = 0. \]
Moving Lines

- A *moving line* is an equation

\[ A(s, t)x + B(s, t)y + C(s, t)z = 0. \]

- A moving line *follows* a parametrization

\[ x = a(s, t), \quad y = b(s, t), \quad z = c(s, t) \]

if \((a(s, t), b(s, t), c(s, t))\) lies on the line \(A(s, t)x + B(s, t)y + C(s, t)z = 0\) for all \(s, t\).
A moving line is an equation
\[ A(s, t)x + B(s, t)y + C(s, t)z = 0. \]

A moving line follows a parametrization
\[ x = a(s, t), \quad y = b(s, t), \quad z = c(s, t) \]
if \((a(s, t), b(s, t), c(s, t))\) lies on the line
\[ A(s, t)x + B(s, t)y + C(s, t)z = 0 \]
for all \(s, t\).

If two moving lines follow a parametrization, their intersection is the parametrization.
Moving Line Picture

Here are two moving lines for an ellipse:
Moving Line Picture

Here are two moving lines for an ellipse:
Here are two moving lines for an ellipse:

Together they \textit{define} the ellipse:
Here are two moving lines for an ellipse:

Together they *define* the *ellipse*:
Here are two moving lines for an ellipse:

Together they *define* the *ellipse*:

This construction is due Steiner in 1832.
Theorem: \( a, b, c \in k[s, t] \) homogeneous, \( \deg = n \), \( \gcd(a, b, c) = 1 \). There exist moving lines \( p, q \) with:

1. Every moving line can be uniquely written as
   \[ up + vq, \]
   where \( u \) and \( v \) are homogeneous of degree \( m - \deg(p) \) and \( m - \deg(q) \).

2. \( \deg(p) + \deg(q) = n \).
**Theorem:** \( a, b, c \in k[s, t] \) homogeneous, \( \deg = n \), \( \gcd(a, b, c) = 1 \). There exist moving lines \( p, q \) with:

1. Every moving line can be uniquely written \( up + vq \), where \( u \) and \( v \) are homogeneous of degree \( m - \deg(p) \) and \( m - \deg(q) \).

2. \( \deg(p) + \deg(q) = n \).

**Definition:** \( p, q \) are a \( \mu \)-basis when \( \mu = \deg(p) \leq \deg(q) = n - \mu \).
Proof

Let \( R = k[s, t] \) and \( I = \langle a, b, c \rangle \subset R \).

The *Hilbert Syzygy Theorem* and a *Hilbert polynomial* computation give a free resolution

\[
0 \rightarrow R(-n - \mu) \oplus R(-2n + \mu) \rightarrow R(-n)^3 \xrightarrow{(a,b,c)} I \rightarrow 0.
\]

The kernel of \((a, b, c)\) is \(\text{Syz}(a, b, c)\). It consists of all triples \((A, B, C)\) \(\in R^3\) such that

\[
Aa + Bb + Cc = 0.
\]

These give the moving lines \(Ax + By + Cz\) that follow the curve.  

QED
Some History

- 1832 – Steiner describes conic sections using moving lines.
Some History

- **1832** – Steiner describes conic sections using moving lines.

- **1887** – Meyer describes the syzygies of $a, b, c \in k[s, t]$ and makes a conjecture for the syzygies of $a_1, \ldots, a_m$. 
Some History

- **1832** – Steiner describes conic sections using moving lines.
- **1887** – Meyer describes the syzygies of $a, b, c \in k[s, t]$ and makes a conjecture for the syzygies of $a_1, \ldots, a_m$. 
- **1890** – Hilbert proves the Syzygy Theorem and proves Meyer’s conjecture.
Some History

- **1832** – Steiner describes conic sections using moving lines.
- **1887** – Meyer describes the syzygies of $a, b, c \in k[s, t]$ and makes a conjecture for the syzygies of $a_1, \ldots, a_m$.
- **1890** – Hilbert proves the Syzygy Theorem and proves Meyer’s conjecture.
- **1995** – Sederberg and Chen interpret moving lines in terms of syzygies.
Moving Curves

Moving lines are not the full story. Let $R = k[s, t]$.

A polynomial

$$F = \sum_{i+j+l=m} A_{ijl}(s, t) x^i y^j z^l \in R[x, y, z]$$

is called a *moving curve* of degree $m$. 
Moving Curves

Moving lines are not the full story. Let $R = k[s, t]$.

- A polynomial

$$F = \sum_{i+j+l=m} A_{ijl}(s, t) x^i y^j z^l \in R[x, y, z]$$

is called a moving curve of degree $m$.

- A moving curve follows our parametrization if

$$\sum_{i+j+l=m} A_{ijl} a^i b^j c^l \equiv 0 \text{ in } R$$
Moving Curves

Moving lines are not the full story. Let $R = k[s, t]$.

- A polynomial

$$F = \sum_{i+j+l=m} A_{ijl}(s, t) \ x^i y^j z^l \in R[x, y, z]$$

is called a moving curve of degree $m$.

- A moving curve follows our parametrization if

$$\sum_{i+j+l=m} A_{ijl} \ a^i b^j c^l \equiv 0 \ in \ R$$

- The moving curve ideal $MC \subset R[x, y, z]$ is generated by these moving curves.
The Rees Algebra

$I = \langle a, b, c \rangle \subset R = k[s, t]$ has Rees algebra

$$R[I] = \bigoplus_{i=0}^{\infty} I^i e^i \subset R[e].$$

Rees algebras are important in commutative algebra.
The Rees Algebra

\[ I = \langle a, b, c \rangle \subset R = k[s, t] \text{ has Rees algebra} \]

\[ R[I] = \bigoplus_{i=0}^{\infty} I^i e^i \subset R[e]. \]

Rees algebras are important in commutative algebra.

The map \((x, y, z) \mapsto (ae, be, ce)\) gives a surjection

\[ R[x, y, z] \longrightarrow R[I]. \]

The kernel is \(MC\). Thus the moving curve ideal gives the defining relations of the Rees algebra!
Example 1

\[ a = 6s^2t^2 - 4t^4, \quad b = 4s^3t - 4st^3, \quad c = s^4 \]

The moving curve ideal has five generators:

- Two moving lines of degree 2 in \( s, t \):
  \[ p = stx + \left(\frac{1}{2}s^2 - t^2\right)y - 2stz \]
  \[ q = s^2x - sty - 2t^2z \]

- Two moving conics of degree 1 in \( s, t \):
  \[ sxy - ty^2 - 2txz - syz + 4tz^2 \]
  \[ s\ x^2 - txy + \frac{1}{2}sy^2 - 2sxz + tyz \]

- The implicit equation:
  \[ y^4 + 4x^3z + 2xy^2z - 16x^2z^2 - 6y^2z^2 + 16xz^3 \]
Example 1 Picture

Here is the curve of Example 1:
Compute Example 1

To generate $MC$, begin with the moving lines:

\[ p = st x + \left( \frac{1}{2} s^2 - t^2 \right) y - 2st z \]
\[ q = s^2 x - st y - 2t^2 z \]

- $s, t^2$ give

\[ p = (tx + \frac{1}{2} sy - 2tz) s + (-y)t^2 \]
\[ q = (sx - ty) s + (-2z)t^2 \]

The **Sylvester form** is

\[
\det \begin{pmatrix} 
  tx + \frac{1}{2} sy - 2tz & -y \\
  sx - ty & -2z 
\end{pmatrix}
\]

This is the first moving conic generator of $MC$!
Compute Example 1

- \( s^2, t \) give the second moving conic generator!
- \( s, t \) and the moving conic generators give
  \[
  (xy - yz)s + (4z^2 - y^2 - 2xz)t \\
  (x^2 + \frac{1}{2}y^2 - 2xz)s + (yz - xy)t
  \]
  The Sylvester form is
  \[
  \det \begin{pmatrix}
  xy - yz & 4z^2 - y^2 - 2xz \\
  x^2 + \frac{1}{2}y^2 - 2xz & yz - xy
  \end{pmatrix}
  \]
  This is the implicit equation!
- The implicit equation is also \( \text{Res}(p, q) \).
Example 2

\[ a = 3s^3t - 3s^2t^2, \quad b = 3s^2t^2 - 3st^3, \quad c = (s^2 + t^2)^2 \]

The moving curve ideal has five generators:

- **Two moving lines** of degree 1,3 in \( s, t \):
  \[ p = tx - sy \]
  \[ q = s^3x + (2s^2t + t^3)y + (3st^2 - 3s^2t)z \]

- **One moving conic** of degree 2 in \( s, t \):
  \[ s^2x^2 + (2s^2 + t^2)y^2 + (3st - 3s^2)yz \]

- **One moving cubic** of degree 1 in \( s, t \):
  \[ sx^3 + 2sxy^2 + ty^3 - 3sxyz + 3sy^2z \]

- **The implicit equation**:
  \[ x^4 + 2x^2y^2 + y^4 - 3x^2yz + 3xy^2z \]
Example 2 Picture

Here is the curve of Example 2:
Compute Example 2

To generate $MC$, begin with the moving lines:

\[ p = tx - sy \]
\[ q = s^3 x + (2s^2 t + t^3)y + (3st^2 - 3s^2t)z \]

- The moving lines give
  \[ p = (-y)s + (x)t \]
  \[ q = (s^2 x)s + ((2s^2 + t^2)y + (3st - 3s^2)z)t \]

The Sylvester form is

\[ \det \begin{pmatrix} -y & x \\ s^2 x & (2s^2 + t^2)y + (3st - 3s^2)z \end{pmatrix} \]

This is $-1$ times the moving conic generator!
Compute Example 2

- \( p = tx - sy \) and the moving conic give

\[
\begin{vmatrix}
-tx^2 + 2sy^2 - 3xyz & ty^2 + 3sy^2 \\
-sx^2 + 2sy^2 - 3xyz & tx^2 + 3sy^2 \\
\end{vmatrix}
\]

This is \(-1\) times the moving cubic generator!

- \( p = tx - sy \) and the moving cubic give

\[
\begin{vmatrix}
-tx^2 + 2sy^2 - 3xyz + 3y^2z & ty^2 + 3sy^2 \\
-sx^2 + 2sy^2 - 3xyz + 3y^2z & tx^2 + 3sy^2 \\
\end{vmatrix}
\]

This is \(-1\) times is the implicit equation!

- The implicit equation is also \( \text{Res}(p, q) \).
Theorems

There are theorems that explain these examples, plus results on parametrized curves in $\mathbb{P}^n$.

Some of the people involved:

C-
C-, Hoffman, and Wang
Busé
Hoon, Simis, and Vasconcelos
Kustin, Polini, and Ulrich
Cortadellas Benítez and D’Andrea
Goldman, Jia, and Wang

I will explain some of this tomorrow.
Rational Plane Curves

**Theorem:** If $C \subset \mathbb{P}^2$ is defined by an irreducible equation of degree $n$, then $C$ is rational $\iff$

$$(n - 1)(n - 2) = \sum_p \nu_p(\nu_p - 1),$$

where the sum is over all singular points $p$ of $C$ and $\nu_p$ is the multiplicity of $C$ at $p$.

**Classical Proof:** We will use *adjoint curves*. A curve $D$ of degree $m$ is *adjoint* to $C$ if:

- At all singular points $p$ of $C$ with multiplicity $\nu_p$, the curve $D$ has multiplicity at least $\nu_p - 1$. 
The Classical Proof

**Lemma:** For \( m \in \{n - 1, n - 2\} \), \( \exists \) a 1-dim linear system of plane curves whose general member \( D \) is adjoint to \( C \) and meets \( C \) in \( mn - (n - 1)(n - 2) - 1 \) fixed smooth points of \( C \).

**Consequences:**
- By Bezout, \( mn = D \cdot C = \sum_p (\nu_p - 1)\nu_p + mn - (n - 1)(n - 2) - 1 + \) one more point.
The Classical Proof

**Lemma:** For \( m \in \{n - 1, n - 2\} \), \( \exists \) a 1-dim linear system of plane curves whose general member \( D \) is adjoint to \( C \) and meets \( C \) in
\[
mn - (n - 1)(n - 2) - 1 \text{ fixed smooth points of } C.
\]

**Consequences:**

- By Bezout, 
  
  \[
  mn = D \cdot C = \sum_p (\nu_p - 1)\nu_p + 
  mn - (n - 1)(n - 2) - 1 + \text{ one more point.}
  \]

- This moving point gives a parametrization!
The Classical Proof

**Lemma:** For $m \in \{n - 1, n - 2\}$, $\exists$ a 1-dim linear system of plane curves whose general member $D$ is adjoint to $C$ and meets $C$ in $mn - (n - 1)(n - 2) - 1$ fixed smooth points of $C$.

**Consequences:**

- By Bezout, $mn = D \cdot C = \sum_p (\nu_p - 1)\nu_p + mn - (n - 1)(n - 2) - 1 +$ one more point.
- This moving point gives a parametrization!
- Linear system $\Rightarrow$ the curve is rational! QED
The Classical Proof

**Lemma:** For \( m \in \{n - 1, n - 2\} \), \( \exists \) a 1-dim linear system of plane curves whose general member \( D \) is adjoint to \( C \) and meets \( C \) in

\[
mn - (n - 1)(n - 2) - 1 \text{ fixed smooth points of } C.
\]

**Consequences:**

- By Bezout, \( mn = D \cdot C = \sum_p (\nu_p - 1)\nu_p + \)
  \[
mn - (n - 1)(n - 2) - 1 + \text{ one more point.}
\]
- This moving point gives a parametrization!
- Linear system \( \Rightarrow \) the curve is rational!  QED

Also: we get the parametrization by a resultant.
Example

Consider the affine curve defined by

\[ F(x, y) = y^4 + 4x^3 + 2xy^2 - 16x^2 - 6y^2 + 16x = 0. \]

Note \((4 - 1)(4 - 2) = 3 \text{ [sing pts]} \cdot 2 \text{ [mult]} \cdot (2 - 1)\).

To parametrize, use the linear system of conics

\[ G_{s,t}(x, y) = sx^2 - txy + \frac{1}{2}sy^2 - 2sx + ty = 0. \]

These adjoint curves all go through the origin.

**Observation:** \(G\) is one of our moving conics!
These conics go through the singular points and the origin, plus one more point that moves.
The Parametrization

Compute the resultants:

\[
\text{Res}(F, G, y) = x(x - 1)^4(x - 2)^2(s^4x - 6t^2s^2 + 4t^4)
\]

\[
\text{Res}(F, G, x) = y^3(y^2 - 2)^2(s^3y - 4ts^2 + 4t^3)
\]

The constant factors show that \( G = 0 \) goes through the origin and the singular points of \( F = 0 \). The other factors give

\[
x = \frac{6s^2t^2 - 4t^4}{s^4}, \quad y = \frac{4s^3t - 4st^3}{s^4}.
\]

This is an affine version of our original parametrization of \( F = 0 \)!
Theorem: Given a proper parametrization of degree $n$, there are elements of the moving curve ideal $MC$ of degree one in $s, t$ and degree $n - 1$ or $n - 2$ in $x, y, z$ can be chosen to be *adjoint linear systems* on the rational curve defined by the parametrization.

- Proof by Busé; $\mu = 1$ by C-, Hoffman, Wang.
- Theorem based on an observation of Sendra.
- Moving lines: small deg $x, y, z$, large deg $s, t$.
  Adjoint curves: large deg $x, y, z$, small deg $s, t$. 

PASI 2009 – p.25/41
Example: The Steiner surface is given by

\[
\begin{align*}
x &= \frac{a(s, t)}{d(s, t)} = \frac{2st}{s^2 + t^2 + 1} \\
y &= \frac{b(s, t)}{d(s, t)} = \frac{2t}{s^2 + t^2 + 1} \\
z &= \frac{c(s, t)}{d(s, t)} = \frac{2s}{s^2 + t^2 + 1}
\end{align*}
\]
Surfaces

Example: The Steiner surface is given by

\[ x = \frac{a(s, t)}{d(s, t)} = \frac{2st}{s^2 + t^2 + 1} \]
\[ y = \frac{b(s, t)}{d(s, t)} = \frac{2t}{s^2 + t^2 + 1} \]
\[ z = \frac{c(s, t)}{d(s, t)} = \frac{2s}{s^2 + t^2 + 1} \]
Boeing 777
Boeing 777

The Boeing 777 was designed using 50 million surface patches.
Guggenheim Bilbao
Gehry Sketch
Bilbao Close-Up
**Affine Case:** A moving plane

\[ Ax + By + Cz + D = 0, \quad A, B, C, D \in k[s, t] \]

follows \( a, b, c, d \) iff \( Aa + Bb + Cc + Dd = 0 \). Thus moving planes live in the syzygy module

\[ \text{Syz}(a, b, c, d) \subseteq k[s, t]^4. \]
**Affine Case:** A moving plane

\[ Ax + By + Cz + D = 0, \quad A, B, C, D \in k[s, t] \]

follows \( a, b, c, d \) iff \( Aa + Bb + Cc + Dd = 0 \). Thus moving planes live in the syzygy module

\[ \text{Syz}(a, b, c, d) \subset k[s, t]^4. \]

**Theorem:** The syzygy module \( \text{Syz}(a, b, c, d) \) is a free \( k[s, t] \)-module of rank 3.
**Affine Case**: A moving plane

\[ Ax + By + Cz + D = 0, \quad A, B, C, D \in k[s, t] \]

follows \( a, b, c, d \) iff \( Aa + Bb + Cc + Dd = 0 \). Thus moving planes live in the syzygy module

\[ \text{Syz}(a, b, c, d) \subset k[s, t]^4. \]

**Theorem**: The syzygy module \( \text{Syz}(a, b, c, d) \) is a free \( k[s, t] \)-module of rank 3.

**Proof**: Auslander-Buchsbaum & Quillen-Suslin!
Projective Case: More complicated!

- \( I = \langle a, b, c, d \rangle \subset R = k[s, t, u] \) homogeneous
- \( \phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3 \) rational map
- **Basepoints** \( V(a, b, c, d) \subset \mathbb{P}^2 \)
- \( S = \text{image} \subset \mathbb{P}^3 \) parametrized surface
- \( \deg \phi \cdot \deg S = n^2 - \sum_p e_p \)
- \( e_p = \text{Hilbert-Samuel Multiplicity} \)
Projective Case

The following are equivalent:

- \( \text{Syz}(a, b, c, d) \) is free
- \( \text{pd}(R/I) = 2 \)
- \( R/I \) is Cohen-Macaulay
- \( I \) is saturated.

**Example:** Cubic surface in \( \mathbb{P}^3 \) has \( a, b, c, d \) deg 3:

- \( \text{Syz}(a, b, c, d) \): 3 moving planes deg 1 in \( s, t, u \).
- Basepoints: Six.

**Also:** No basepoints \( \Rightarrow \) \( \text{Syz}(a, b, c, d) \) not free.
The Bihomogeneous Case

Geometric Modeling often uses *rectangular* surfaces patches, built from polynomials in $s, t$ whose Newton polygon is a rectangle.

This leads naturally to a parametrization

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$$

(assuming no basepoints), where $\phi$ is given by *bihomogeneous* polynomials of bidegree $(n, m)$.

Bigraded commutative algebra is *very different*!

I will give an example on Thursday.
The Affine Case

A basis of $\text{Syz}(a, b, c, d)$ over $k[s, t]$ is a $\mu$-basis.
The Affine Case

A basis of $\text{Syz}(a, b, c, d)$ over $k[s, t]$ is a $\mu$-basis.

Write the $\mu$-basis as

\[
\begin{align*}
p &= A \, x + B \, y + C \, z + D &= 0 \\
q &= A' \, x + B' \, y + C' \, z + D' &= 0 \\
r &= A'' \, x + B'' \, y + C'' \, z + D'' &= 0
\end{align*}
\]
The Affine Case

A basis of $\text{Syz}(a, b, c, d)$ over $k[s, t]$ is a $\mu$-basis.

Write the $\mu$-basis as

\[
\begin{align*}
p &= Ax + By + Cz + D = 0 \\
q &= A'x + B'y + C'z + D' = 0 \\
r &= A''x + B''y + C''z + D'' = 0
\end{align*}
\]

By Cramer, $a, b, c, d$ are the $3 \times 3$ minors of

\[
\begin{pmatrix}
p \\ q \\ r
\end{pmatrix} = \begin{pmatrix}
A & B & C & D \\
A' & B' & C' & D' \\
A'' & B'' & C'' & D''
\end{pmatrix}
\]
Resultant of a $\mu$-Basis

For surfaces, the resultant of an affine $\mu$-basis *almost* gives the implicit equation.
Resultant of a $\mu$-Basis

For surfaces, the resultant of an affine $\mu$-basis *almost* gives the implicit equation.

**Analysis**: At a point $(x, y, z)$ where

$$\text{Res}(p, q, r) = 0,$$

the equations

\[
\begin{align*}
p &= Ax + By + Cz + D = 0 \\
q &= A'x + B'y + C'z + D' = 0 \\
r &= A''x + B''y + C''z + D'' = 0
\end{align*}
\]

have a solution $s, t$ (possibly at $\infty$).
No Basepoints

\[
\begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
= \begin{pmatrix}
A & B & C & D \\
A' & B' & C' & D' \\
A'' & B'' & C'' & D''
\end{pmatrix}
\]

has rank 3 since \(a, b, c, d\) are the \(3 \times 3\) minors. So no basepoints \(\Rightarrow\) the moving planes always have a unique point of intersection!
Basepoints

At a basepoint, the parameter values “blow up” to an exceptional curve on the surface. These curves come in three flavors:

- A line.
- A plane curve.
- A space curve.
Basepoints

At a basepoint, the parameter values “blow up” to an exceptional curve on the surface. These curves come in three flavors:

- A line.
- A plane curve.
- A space curve.

These cases correspond to the rank of

\[
\begin{pmatrix}
p \\ q \\ r
\end{pmatrix} = \begin{pmatrix}
A & B & C & D \\
A' & B' & C' & D' \\
A'' & B'' & C'' & D''
\end{pmatrix}.
\]
Rank 2 Basepoints

Here, the moving planes intersect in a line:

\[ q = 0 \]
\[ p = 0 \]
\[ r = 0 \]

Furthermore:

- The resultant \( \text{Res}(p, q, r) \) vanishes exactly on the surface, at least for \( s, t \) finite.
- A basepoint has rank two \( \iff \) it is LCI!
Rank 1 Basepoints

Here, the moving planes coincide:

Furthermore:

- The resultant $\text{Res}(p, q, r)$ has an extraneous factor $= \text{the equation of the plane to the power } e_p - d_p$, $d_p = \dim_k \mathcal{O}_p/\langle a, b, c, d \rangle$.

- A basepoint has rank one $\iff \langle a, b, c, d \rangle$ is almost LCI.
Rank 0 Basepoints

Here, the moving “planes” are the ambient space, since we have a space curve. Thus:

- The resultant $\text{Res}(p, q, r)$ vanishes identically.
- A basepoint has rank zero $\iff$ locally $\langle a, b, c, d \rangle$ requires four generators.

Hence

$$\text{Res}(p, q, r)$$

requires a truly bad basepoint before it vanishes!