Correction to Proposition 5.5.4

Proposition 5.5.4 on page 98 of *Mirror Symmetry and Algebraic Geometry* is incorrect as stated. The purpose of this note is to give the correct statement, along with details of the proof. You will see below that three major changes are needed in Proposition 5.5.4:

- First, in equation (5.43) on page 98, we need to replace λ_{r+i} with $-\lambda_{r+i}$ in the definition of q_j.
- Second, in Proposition 5.5.4, we need to replace $\tilde{I}$ with $\big(\prod_{i=1}^k \lambda_{r+i}^{-1}\big)\tilde{I}$ in order to satisfy the GKZ system.
- Third, in Proposition 5.5.4, we need to replace the vector $\hat{\beta} = \vec{0}$ with $\hat{\beta} = (0,\dots,0,-1,\dots,-1)$.

The following text explains how the exposition on pages 98 and 99 needs to be changed in order to state and prove a correct version of Proposition 5.5.4.

As on page 98, we put $\overline{T} = \overline{N} \otimes \mathbb{C}^*$. Then consider the $\overline{T}$-invariant variables

$$q_j = \Big(\prod_\rho \lambda_\rho^{D_\rho\cdot\beta_j}\Big)\Big/\Big(\prod_{i=1}^k (-\lambda_{r+i})^{c_1(\mathcal{L}_i)\cdot\beta_j}\Big), \qquad 1 \le j \le r-n. \tag{1}$$

Note that this differs from (5.43) on page 98 because of the minus signs. We will see that these signs are essential.

As on page 98, we let $T_1,\dots,T_{r-n}$ be dual to $\beta_1,\dots,\beta_{r-n}$ and we define

$$q^\beta = \prod_j q_j^{T_j\cdot\beta} \quad \text{for } \beta \in M(X)_\mathbb{Z} \subset H_2(X,\mathbb{Z}).$$

It is then easy to see that

$$q^\beta = \Big(\prod_{i=1}^k (-1)^{c_1(\mathcal{L}_i)\cdot\beta}\Big)\Big(\prod_\rho \lambda_\rho^{D_\rho\cdot\beta}\Big)\Big/\Big(\prod_{i=1}^k \lambda_{r+i}^{c_1(\mathcal{L}_i)\cdot\beta}\Big). \tag{2}$$

We also put $t_j = \log q_j$, so that $q_j = e^{t_j}$.

We have the cohomology-valued expression

$$\tilde{I} = e^{\sum_j t_j T_j} \sum_{\beta\in M(X)_\mathbb{Z}} q^\beta \frac{\Big(\prod_{i=1}^k \prod_{m=1}^{c_1(\mathcal{L}_i)\cdot\beta} (c_1(\mathcal{L}_i)+m)\Big)\Big(\prod_\rho \prod_{m=-\infty}^0 (D_\rho+m)\Big)}{\prod_\rho \prod_{m=-\infty}^{D_\rho\cdot\beta} (D_\rho+m)}.$$

Note that $c_1(\mathcal{L}_i)\cdot\beta \ge 0$ since $\mathcal{L}_i$ is generated by global sections. This fact wasn't used in the proof given on pages 99–100. By using $c_1(\mathcal{L}_i)\cdot\beta \ge 0$, we will be able to simplify the argument a bit.

Here is the correct version of Proposition 5.5.4.

PROPOSITION 5.5.4. *The formal function $\big(\prod_{i=1}^k \lambda_{r+i}^{-1}\big)\tilde{I}$ satisfies the $\mathcal{A}$-system associated to (5.42) with $\hat{\beta} = (0,\dots,0,-1,\dots,-1) \in N \times \mathbb{Z}^k$.*

Before beginning the proof, note that the factor $\prod_{i=1}^k \lambda_{r+i}^{-1}$ makes sense if you read page 94. There, we used $\Box_\ell \lambda_0^{-1}$ in order to derive the 5th order equation (5.37). Also, the new version of $\hat{\beta}$ is consistent with the discussion following (5.31) on page 91. Finally, the minus sign in (1) is consistent with the discussion of signs in Section 6.3.3 on pages 155–156.

Proof. In the statement of this proposition, $\hat{\beta}$ of course refers to the vector needed to define the operators Z_i in (5.27). First suppose that $\hat{\beta} = \vec{0}$, and let Z_i be the corresponding operators for this $\hat{\beta}$. Then, as we discussed in Section 5.5.2, the assertion that $Z_i(\tilde{I}) = 0$ for all i amounts to the $\overline{T}$-invariance of the formal function $\tilde{I}$. This follows immediately, since the q_j were constructed to be $\overline{T}$-invariant, and the λ_i appear in $\tilde{I}$ only through the q_j. Finally, if we now switch to the Z_j for $\hat{\beta} = (0, \ldots, 0, -1, \ldots, -1) \in N \times \mathbb{Z}^k$, we see that the Z_j vanish on $\big(\prod_{i=1}^k \lambda_{r+i}^{-1}\big)\, \tilde{I}$.

We now consider the operators $\Box_\beta$ for $\beta \in H_2(X, \mathbb{Z}) \simeq \Lambda$. By Lemma 5.5.3, β gives the relation $\sum_\rho (D_\rho \cdot \beta) w_\rho - \sum_i (c_1(\mathcal{L}_i) \cdot \beta) u_i = 0$. We observed above that $c_1(\mathcal{L}_i) \cdot \beta \geq 0$. It follows that $\Box_\beta$ can be written as

$$\Box_\beta = \prod_{D_\rho \cdot \beta > 0} \partial_\rho^{D_\rho \cdot \beta} - \prod_{D_\rho \cdot \beta < 0} \partial_\rho^{-D_\rho \cdot \beta} \prod_{i=1}^k \partial_{r+i}^{c_1(\mathcal{L}_i) \cdot \beta}, \tag{3}$$

where $\partial_\rho = \partial/\partial\lambda_\rho$ and $\partial_{r+i} = \partial/\partial\lambda_{r+i}$. We need to prove $\Box_\beta\big(\prod_{i=1}^k \lambda_{r+i}^{-1}\big)\, \tilde{I} = 0$.

As usual, we put $\delta_\rho = \lambda_\rho \partial_\rho$ and $\delta_{r+i} = \lambda_{r+i}\partial_{r+i}$. We also put $\delta_{q_j} = q_j \partial/\partial q_j = \partial/\partial t_j$. Our methods applied to (1) give the identities

$$\begin{aligned} \delta_\rho &= \textstyle\sum_j (D_\rho \cdot \beta_j) \delta_{q_j} \\ \delta_{r+i} &= -\textstyle\sum_j (c_1(\mathcal{L}_i) \cdot \beta_j) \delta_{q_j} \end{aligned} \tag{4}$$

when applied to $\overline{T}$-invariant functions. We also compute that

$$\delta_{q_j}\left(e^{\sum_i t_i T_i} q^\beta\right) = (T_j + T_j \cdot \beta)\left(e^{\sum_i t_i T_i} q^\beta\right). \tag{5}$$

Combining (4) and (5), we obtain

$$\begin{aligned} \delta_\rho\left(e^{\sum_i t_i T_i} q^\beta\right) &= \big(D_\rho + D_\rho \cdot \beta\big)\left(e^{\sum_i t_i T_i} q^\beta\right) \\ \delta_{r+i}\left(e^{\sum_i t_i T_i} q^\beta\right) &= -\big(c_1(\mathcal{L}_i) + c_1(\mathcal{L}_i) \cdot \beta\big)\left(e^{\sum_i t_i T_i} q^\beta\right). \end{aligned} \tag{6}$$

In (6), we have used the identity of divisors $D = \sum_j (D \cdot \beta_j)\, T_j$ for $D = D_\rho$ and $D = c_1(\mathcal{L}_i)$. This identity follows because T_j is the dual basis to β_j.

The operator $\Box_\beta$ is defined in terms of ∂_ρ and ∂_{r+i}. In order to write this in terms of $\delta_\rho = \lambda_\rho \partial_\rho$ and $\delta_{r+i} = \lambda_{r+i}\partial_{r+i}$, we will consider the operator

$$\Box'_\beta \;=\; \prod_{D_\rho \cdot \beta > 0} \lambda_\rho^{D_\rho \cdot \beta} \;\prod_{i=1}^k \lambda_{r+i}\;\; \Box_\beta \;\prod_{i=1}^k \lambda_{r+i}^{-1}.$$

It suffices to show that $\tilde{I}$ is annihilated by $\Box'_\beta$. By (2), we know that

$$\prod_{D_\rho \cdot \beta > 0} \lambda_\rho^{D_\rho \cdot \beta} = q^\beta \Big(\prod_{i=1}^k (-1)^{c_1(\mathcal{L}_i) \cdot \beta}\Big)\Big(\prod_{D_\rho \cdot \beta < 0} \lambda_\rho^{-D_\rho \cdot \beta}\Big)\Big(\prod_{i=1}^k \lambda_{r+i}^{c_1(\mathcal{L}_i) \cdot \beta}\Big).$$

Combining this with (3), it follows that

$$\begin{aligned}\Box'_\beta = {} & \prod_{D_\rho\cdot\beta>0} \lambda_\rho^{D_\rho\cdot\beta}\partial_\rho^{D_\rho\cdot\beta} \;- \\ & q^\beta \prod_{i=1}^{k}(-1)^{c_1(\mathcal{L}_i)\cdot\beta} \prod_{D_\rho\cdot\beta<0}\Big(\lambda_\rho^{-D_\rho\cdot\beta}\partial_\rho^{-D_\rho\cdot\beta}\Big)\prod_{i=1}^{k}\Big(\lambda_{r+i}^{c_1(\mathcal{L}_i)\cdot\beta+1}\partial_{r+i}^{c_1(\mathcal{L}_i)\cdot\beta}\lambda_{r+i}^{-1}\Big).\end{aligned}$$

Then, using (5.35), we compute that

$$\begin{aligned}\lambda_\rho^m\partial_\rho^m &= \delta_\rho(\delta_\rho-1)\cdots(\delta_\rho-m+1)\\ \lambda_{r+i}^{m+1}\partial_{r+i}^m\lambda_{r+i}^{-1} &= (\delta_{r+i}-1)\cdots(\delta_{r+i}-m).\end{aligned}$$

Hence the above formula for $\Box'_\beta$ can be written

$$\Box'_\beta = A - q^\beta B,$$

where

$$A = \prod_{D_\rho\cdot\beta>0}\delta_\rho(\delta_\rho-1)\cdots(\delta_\rho-D_\rho\cdot\beta+1) \tag{7}$$

and

$$\begin{aligned} B \;=\; & \prod_{i=1}^{k}(-1)^{c_1(\mathcal{L}_i)\cdot\beta}\prod_{D_\rho\cdot\beta<0}\delta_\rho(\delta_\rho-1)\cdots(\delta_\rho+D_\rho\cdot\beta+1)\times \\ & \prod_{i=1}^{k}(\delta_{r+i}-1)\cdots(\delta_{r+i}-c_1(\mathcal{L}_i)\cdot\beta).\end{aligned} \tag{8}$$

We now apply the operators A and B to $e^{\sum_i t_iT_i}q^{\beta'}$. By (6) and (7), we get

$$\begin{aligned} & A\big(e^{\sum_i t_iT_i}q^{\beta'}\big) = e^{\sum_i t_iT_i}q^{\beta'}\times \\ & \prod_{D_\rho\cdot\beta>0}(D_\rho+D_\rho\cdot\beta')(D_\rho+D_\rho\cdot\beta'-1)\cdots(D_\rho+D_\rho\cdot(\beta'-\beta)+1),\end{aligned} \tag{9}$$

and by (6) and (8), we also obtain

$$\begin{aligned} & B\big(e^{\sum_i t_iT_i}q^{\beta''}\big) = e^{\sum_i t_iT_i}q^{\beta''}\prod_{i=1}^{k}(-1)^{c_1(\mathcal{L}_i)\cdot\beta}\times \\ & \prod_{D_\rho\cdot\beta<0}(D_\rho+D_\rho\cdot\beta'')(D_\rho+D_\rho\cdot\beta''-1)\cdots(D_\rho+D_\rho\cdot(\beta''+\beta)+1)\times \\ & \prod_{i=1}^{k}\big(-(c_1(\mathcal{L}_i)+c_1(\mathcal{L}_i)\cdot\beta''+1)\big)\cdots\big(-(c_1(\mathcal{L}_i)+c_1(\mathcal{L}_i)\cdot(\beta''+\beta))\big).\end{aligned}$$

Sign cancellations allow us to write this as

$$\begin{aligned} & B\big(e^{\sum_i t_iT_i}q^{\beta''}\big) = e^{\sum_i t_iT_i}q^{\beta''}\times \\ & \prod_{D_\rho\cdot\beta<0}(D_\rho+D_\rho\cdot\beta'')(D_\rho+D_\rho\cdot\beta''-1)\cdots(D_\rho+D_\rho\cdot(\beta''+\beta)+1)\times \\ & \prod_{i=1}^{k}(c_1(\mathcal{L}_i)+c_1(\mathcal{L}_i)\cdot\beta''+1)\cdots(c_1(\mathcal{L}_i)+c_1(\mathcal{L}_i)\cdot(\beta''+\beta)).\end{aligned} \tag{10}$$

The reason for using β' in (9) and β'' in (10) will soon become clear.

Before we apply A and B to $\tilde{I}$, we need some notation. Using the fixed β which is used in the definition of $\Box_\beta$ and $\Box'_\beta$, we define

$$C(\beta') = \prod_{i=1}^{k} \prod_{m=1}^{c_1(\mathcal{L}_i)\cdot\beta'} (c_1(\mathcal{L}_i)+m)$$

$$S(\beta') = \prod_{D_\rho\cdot\beta>0} \frac{\prod_{m=-\infty}^{0}(D_\rho+m)}{\prod_{m=-\infty}^{D_\rho\cdot\beta'}(D_\rho+m)}$$

$$T(\beta') = \prod_{D_\rho\cdot\beta=0} \frac{\prod_{m=-\infty}^{0}(D_\rho+m)}{\prod_{m=-\infty}^{D_\rho\cdot\beta'}(D_\rho+m)}$$

$$U(\beta') = \prod_{D_\rho\cdot\beta<0} \frac{\prod_{m=-\infty}^{0}(D_\rho+m)}{\prod_{m=-\infty}^{D_\rho\cdot\beta'}(D_\rho+m)}.$$

In terms of these formulas, $\tilde{I}$ can be written

$$\tilde{I} = e^{\sum t_j T_j} \sum_{\beta'} q^{\beta'} C(\beta')\, S(\beta')\, T(\beta')\, U(\beta'). \tag{11}$$

If we now apply A to (11) and use (9), we easily obtain

$$A\,\tilde{I} = e^{\sum t_j T_j} \sum_{\beta'} q^{\beta'} C(\beta')\, S(\beta'-\beta)\, T(\beta')\, U(\beta') \tag{12}$$

since one easily verifies that

$$\left(\prod_{D_\rho\cdot\beta>0}\prod_{m=D_\rho\cdot(\beta'-\beta)+1}^{D_\rho\cdot\beta'}(D_\rho+m)\right) S(\beta') = S(\beta'-\beta).$$

Furthermore, if we write (11) in terms of β'' instead of β' and apply B using (10), then we obtain

$$q^\beta B\,\tilde{I} = e^{\sum t_j T_j} \sum_{\beta''} q^{\beta''+\beta} C(\beta''+\beta)\, S(\beta'')\, T(\beta'')\, U(\beta''+\beta) \tag{13}$$

because of the identities

$$\left(\prod_{i=1}^{k}\prod_{m=c_1(\mathcal{L}_i)\cdot\beta+1}^{c_1(\mathcal{L}_i)\cdot(\beta''+\beta)}(c_1(\mathcal{L}_i)+m)\right) C(\beta'') = C(\beta''+\beta)$$

and

$$\left(\prod_{D_\rho\cdot\beta<0}\prod_{m=D_\rho\cdot(\beta''+\beta)+1}^{D_\rho\cdot\beta''}(D_\rho+m)\right) U(\beta'') = U(\beta''+\beta).$$

The final observation is that $T(\beta'') = T(\beta''+\beta)$ by the definition of T. It follows that we can write (13) as

$$q^\beta B\,\tilde{I} = e^{\sum t_j T_j} \sum_{\beta''} q^{\beta''+\beta} C(\beta''+\beta)\, S(\beta'')\, T(\beta''+\beta)\, U(\beta''+\beta). \tag{14}$$

In the expressions (12) for $A\,\tilde{I}$ and (14) for $q^\beta B\,\tilde{I}$, note that the term in (12) for $\beta' \in M(X)_{\mathbb{Z}}$ equals the term in (14) containing $\beta'' \in M(X)_{\mathbb{Z}}$ provided $\beta' = \beta''+\beta$. It follows that these terms cancel in

$$\Box'_\beta \tilde{I} = A\,\tilde{I} - q^\beta B\,\tilde{I}.$$

After this cancelation, we are left with two classes of terms: those in the first sum corresponding to $\beta' \in M(X)_{\mathbb{Z}}$ such that $\beta' - \beta \notin M(X)_{\mathbb{Z}}$, and those in the second sum corresponding to $\beta'' \in M(X)_{\mathbb{Z}}$ such that $\beta'' + \beta \notin M(X)_{\mathbb{Z}}$. The argument given on page 100 applies without change, so that these two classes of terms vanish. This completes the proof of the theorem. □

Here are two further comments:

- Page 100: The final three lines of page 100, which discuss $\hat{\beta} = \vec{0}$ versus $\hat{\beta} = (0, \ldots, 0, -1)$, can now be omitted since for the hypersurface case, the $\hat{\beta}$ of Proposition 5.5.4 now agrees with $\hat{\beta} = (0, \ldots, 0, -1)$ on page 91.
- Page 101: In the discussion at the top of the page concerning the quintic threefold, we need to point out that $\tilde{I}$ is a function of q_1, while on pages 93–94, we used the variable z defined in (5.36). These variables are related by
$$q_1 = -z$$
by the definition of q_1 given in (1). So one has to make this variable change before asserting that $\tilde{I}$ satisfies (5.37).

One way to see the second bullet is to note that for the quintic threefold, (5.45) shows that $\tilde{I}$ is given by the formula

$$\tilde{I} = e^{t_1 H} \sum_{d=0}^{\infty} e^{dt_1} \frac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}.$$

Expanding this, one finds that the degree 0 part of $\tilde{I}$ is

$$\sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5} e^{dt_1} = \sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5} q_1^d$$

since $q_1 = e^{t_1}$. In contrast, the holomorphic solution of (5.37) and (5.38) is the formula

$$y_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} (-1)^n z^n$$

from (2.23) in Chapter 2 (see also the top of page 95). So we clearly need $q_1 = -z$.

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